VANDERBILT UNIVERSITY, ANALYSIS STUDY GUIDE

Question 1. Let $\{a_i\}_{i=1}^n$ be a finite collection of points in $\mathbb{R}^n$. Let μ be a Radon measure in $\mathbb{R}^n$ whose support is contained in $\cup_{i=1}^n a_i$. Prove that μ is a linear combination of the measures δ_{a_i}, where δ_{a_i} is the Dirac measure at a_i.

Question 2. Let μ^* be the Lebesgue outer measure on $\mathbb{R}^n$. Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of subsets of $\mathbb{R}^n$. Let $\mathscr{A} \subseteq \mathscr{P}(\mathbb{R}^n)$ be the set of subsets $A \in \mathscr{P}(\mathbb{R}^n)$ such that, for every $E \subseteq \mathbb{R}^n$, it holds that

$$\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(A^c \cap E),$$

where A^c is the complement of A, i.e., $A^c = \mathbb{R}^n \backslash A$. Prove that $\mathscr{A}$ is a σ-algebra on $\mathbb{R}^n$.

Question 3. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ. Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a function. Let $B \subseteq U$ be measurable set, and $A \subset U$ a negligible set (i.e., a set of zero measure).
(a) Prove that if f is a Lipschitz map, then $f(A)$ is negligible.
(b) Prove that if f is a C^1 map, then $f(A)$ is negligible.
(c) Prove that if f is a C^1 map, then $f(B)$ is measurable.

Question 4. State and prove Carathédory's criterion for determining when a measure is Borel.

Question 5. Let μ be a Borel measure on $\mathbb{R}^n$ that is finite on compact sets. Define $f : \mathbb{R}^n \to \mathbb{R}$ as follows: for any $x \in \mathbb{R}^n$, let $f(x) = \mu(B_1(x))$, where $B_1(x)$ is the open ball of radius one centered at x. Prove that f attains its infimum on every compact set.

Question 6. Let μ be the Lebesgue measure on $\mathbb{R}$ and $A \subset \mathbb{R}$ a Lebesgue measurable set of finite measure. Define $f : \mathbb{R} \to [0, \infty)$ by $f(x) = \mu(A \cap (-\infty, x])$. Prove that f is continuous.

Question 7. Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a function. Suppose that there exist a (not necessarily Lebesgue measurable) set $A \subseteq [a, b]$ and a constant $C > 0$ such that f is differentiable at every $x \in A$ and

$$|f'(x)| \leq C,$$

for every $x \in A$. Prove that

$$\mu^*(f(A)) \leq C\mu^*(A),$$

where μ^* is the Lebesgue outer measure on $\mathbb{R}$.

Question 8. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set satisfying $\mu(A) > 0$, where μ is the Lebesgue measure. Prove that given $\varepsilon > 0$, there exists a bounded interval $I_\varepsilon = [a, b]$ with $a < b$ such that

$$\mu(A \cap I_\varepsilon) \geq (1 - \varepsilon)\mu(I_\varepsilon).$$

Question 9. Let μ be a Radon measure on $\mathbb{R}^n$ and define $f : (0, \infty) \to [0, \infty]$ by

$$f(r) = \sup_{x \in \mathbb{R}^n} \mu(B_r(x)),$$

where $B_r(x)$ is the open ball of radius r centered at x. Assume that f is $\mathbb{R}$-valued and that

$$\liminf_{r \to \infty} \frac{f(r)}{r^n} = 0.$$

Prove that $\mu = 0$.

Question 10. State the definition of the s-dimensional Hausdorff measure on $\mathbb{R}^n$, $0 \leq s \leq n$, and prove that the Hausdorff measure is a Borel measure.

Question 11. Let $f : [a, b] \to \mathbb{R}^n$ be a continuous map and denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure on $\mathbb{R}^n$. Prove that

$$|f(a) - f(b)| \leq \mathcal{H}^1(f([a, b])).$$

Question 12. Let $f : [a, b] \to \mathbb{R}^n$ be an injective Lipschitz map and denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure on $\mathbb{R}^n$. Prove that $\mathcal{H}^1(f([a, b]))$ is finite.

Question 13. Prove that for every connected set $A \subseteq \mathbb{R}^n$, it holds that $\mathcal{H}^1(A) \geq \operatorname{diam}(A)$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure on $\mathbb{R}^n$ and $\operatorname{diam}(A)$ is the diameter of A.

Question 14. Let μ^* be the Lebesgue outer measure on $\mathbb{R}^n$ and denote by $\int^* f\, d\mu$ the corresponding upper integral of a non-negative real valued function f. Let V be a non-empty set of real valued non-negative lower semi-continuous functions on $\mathbb{R}^n$. Assume that V is directed with respect to the relation $\leq$. Prove that

$$\int^* \sup_{f \in V} f\, d\mu = \sup_{f \in V} \int^* f\, d\mu.$$

Provide a counter-example showing that the result is not necessarily true if we do not assume the functions in V to be lower semi-continuous.

Question 15. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ. Let $\{g_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be sequences of non-negative real valued integrable functions on $\mathbb{R}^n$, such that $\{g_n\}_{n=1}^{\infty}$ converges a.e. to an integrable function g, and $\{f_n\}_{n=1}^{\infty}$ converges a.e. to a function f. Assume that for every n, $0 \leq f_n \leq g_n$ a.e. Suppose further that

$$\lim_{n \to \infty} \int g_n\, d\mu = \int g\, d\mu.$$

Prove that f is integrable and that

$$\lim_{n \to \infty} \int f_n\, d\mu = \int f\, d\mu.$$

Question 16. State and prove the monotone convergence theorem.

Question 17. State and prove Fatou's lemma.

Question 18. State and prove the dominated convergence theorem.

Question 19. Show that the dominated convergence theorem is not true for nets of functions.

Question 20. All measure related statements in this problem refer to the Lebesgue measure. Find a bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ such that there does not exist any sequence of continuous functions converging to f in $L^\infty(\mathbb{R})$.

Question 21. Consider the space $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, defined with respect to the Lebesgue measure, and denote the corresponding norm by $\| \cdot \|_p$. Let $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R}^n)$ be a sequence of functions such that, for some function f,

$$\lim_{n\to\infty} \| f_n \|_p = \| f \|_p,$$

and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e. in } \mathbb{R}^n.$$

Prove or give a counter-example: $\{f_n\}_{n=1}^{\infty}$ converges to f in L^p.

Question 22. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ. Let $f \in L^1(\mathbb{R}^n)$, and set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu.$$

(a) Show that Mf is lower semi-continuous and that the set $A_\lambda = \{x \in \mathbb{R}^n \,|\, |(Mf)(x)| > \lambda\}$ is open for each $\lambda > 0$.

(b) Prove that there exists a compact set $K \subseteq A_\lambda$ such that $2\mu(K) \geq \mu(A_\lambda)$, and that for each $x \in K$ there exists a ball $B_\rho(x)$, where ρ depends on x, such that

$$\frac{1}{\mu(B_\rho(x))} \int_{B_\rho(x)} |f| \, d\mu > \lambda.$$

Show that there exist finitely many $\{B_\rho(x_i)\}_{i=1}^{N}$ of the these balls that are pair-wise disjoint and such that $\{B_{3\rho}(x_i)\}_{i=1}^{N}$ covers K.

(c) Use parts (a) and (b) to conclude that

$$\mu(A_\lambda) \leq \frac{2 \cdot 3^n}{\lambda} N_1(f).$$

Question 23. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ. Let $f \in L^1(\mathbb{R}^n)$ and $K \subset \mathbb{R}^n$ be a compact set. Prove that

$$\lim_{|x|\to\infty} \int_{x+K} |f| \, d\mu = 0.$$

Question 24. All measure related statements in this problem refer to the Lebesgue measure. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous functions and assume that $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$. Prove that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Question 25. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ. Let $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R})$ be a sequence such that $\{f_n\}_{n=1}^{\infty}$ converges almost everywhere to a function f. Assume that for every $\varepsilon > 0$, there exist a measurable set $A \subseteq \mathbb{R}$, a non-negative function $h \in L^1(\mathbb{R})$, and an integer $N \geq 1$ such that

$$\int_{A^c} |f_n| \, d\mu \leq \varepsilon$$

for every $n \geq N$, and $|f_n(x)| \leq h(x)$ for every $x \in A$ and every $n \geq N$ (A^c is the complement of A). Prove that $f \in L^1(\mathbb{R})$ and that f_n converges to f in $L^1(\mathbb{R})$.

Question 26. All measure related statements in this problem refer to the Lebesgue measure. Find a sequence of functions in $L^p((0,1))$, with $1 \leq p < \infty$, that converges weakly to zero but does not converge to zero in $L^p((0,1))$.

Question 27. Let X be a locally compact Hausdorff topological space. Denote by $\mathscr{K}(X;\mathbb{C})$ the space of complex valued continuous compactly supported functions on X. Denote by $\mathscr{K}(X,A;\mathbb{C})$ the space of all $f \in \mathscr{K}(X;\mathbb{C})$ such that $\operatorname{supp}(f) \subseteq A$, where $\operatorname{supp}(f)$ denotes the support of f. For each compact set $K \subseteq X$, endow $\mathscr{K}(X,K;\mathbb{C})$ with the topology of uniform convergence. Endow $\mathscr{K}(X;\mathbb{C})$ with the inductive limit of locally convex topologies given by $\mathscr{K}(X,K;\mathbb{C})$ as K ranges over all compact sets of X.

(a) Prove that a linear form μ on $\mathscr{K}(X;\mathbb{C})$ defines a complex Radon measure on X if and only if for each $K \subseteq X$, there exists a constant M_K such that for every $f \in \mathscr{K}(X;\mathbb{C})$ with $\operatorname{supp}(f) \subseteq K$, we have

$$|\mu(f)| \leq M_K \sup_{x\in X} |f(x)|,$$

where $|\cdot|$ is the absolute value in $\mathbb{C}$.

(b) State the definition of a positive Radon measure.

(c) Let $\mathscr{K}(X;\mathbb{R})$ be defined as in (a), but with $\mathbb{C}$ replaced by $\mathbb{R}$. Prove that any positive linear form on $\mathscr{K}(X;\mathbb{R})$ defines a Radon measure on X.

Question 28. Let X be a locally compact Hausdorff topological space and μ a complex Radon measure on X.

(a) State the definition of the restriction of μ to an open set U of X.

(b) Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of X. Suppose that for each $\alpha \in A$, we are given a measure μ_α on U_α. Assume that for each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the restrictions of μ_α and μ_β to $U_\alpha \cap U_\beta$ agree. Prove that there exists a unique measure μ on X such that $\mu|_{U_\alpha} = \mu_\alpha$ for each $\alpha \in A$.

Question 29. Let X be a locally compact Hausdorff topological space. Prove that every real Radon measure on X is the difference of two positive Radon measures.

Question 30. Let X be a locally compact metric space. Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of Radon measures on X. Prove that $\{\mu_n\}_{n=1}^\infty$ converges in the vague topology to a Radon measure μ if and only if $\mu_n(A) \to \mu(A)$ for all Borel sets $A \subseteq X$ that are contained in a compact set and that satisfy $\mu(\partial A) = 0$.

Question 31. Let X be a locally compact Hausdorff topological space and μ a Radon measure on X.

(a) State the definition of an integrable set $A \subseteq X$.

(b) Let $\{A_n\}_{n=1}^\infty$ be a decreasing sequence of integrable sets. Prove that

$$\mu(\bigcap_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n).$$

(c) Let $\{A_n\}_{n=1}^\infty$ be an increasing sequence of integrable sets. Prove that $\cup_{n=1}^\infty A_n$ is integrable if and only if $\sup_n \mu(A_n) < \infty$, and in this case

$$\mu(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n).$$

(d) Let $\mathcal{G}$ be a family of integrable closed sets directed with respect to the relation $\supseteq$. Prove that

$$A = \bigcap_{G \in \mathcal{G}} G$$

is integrable, and that

$$\mu(A) = \inf_{G \in \mathcal{G}} \mu(G).$$

Question 32. Let X be a locally compact Hausdorff topological space and μ a Radon measure on X.

(a) State the definition of an integrable set $A \subseteq X$.

(b) Show that a set A is integrable if and only if for every $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $\mu^*(A \backslash K) \leq \varepsilon$, where μ^* is the outer measure canonically associated with μ.

Question 33. Let X be a locally compact σ-compact Hausdorff topological space and μ a Radon measure on X.

(a) Give the definition of $L^\infty(X)$.

(b) Prove that $L^\infty(X)$ is complete.

Question 34. Let X be a locally compact σ-compact Hausdorff topological space and μ a Radon measure on X. Let $g \geq 0$ be a locally integrable function. Set $\nu = g\mu$. Prove that $f : X \to \overline{\mathbb{R}}$ is ν-integrable if and only if fg is μ-integrable, in which case

$$\int f \, d\nu = \int fg \, d\mu.$$

Question 35. Let X be a locally compact σ-compact Hausdorff space. Let μ, λ, and ν be Radon measures on X. Suppose that every μ-measurable set is λ-measurable and ν-measurable, that every μ-negligible set is λ-negligible and ν-negligible, and that $\lambda(X) = 1 = \nu(X)$.

(a) Let Σ be the collection of all μ-measurable sets. Explain why the quantity

$$\sup_{A \in \Sigma} |\lambda(A) - \nu(A)|$$

is a well-defined real number.

(b) Prove there there exist μ-integrable functions f and g, whose equivalence classes are uniquely determined by λ and ν, respectively, such that

$$\sup_{A \in \Sigma} |\lambda(A) - \nu(A)| = \frac{1}{2} \int |f - g| \, d\mu.$$

Question 36. State Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions in $\mathbb{R}^n$, and prove the result in the case $n = 1$.

Question 37. All measure related statements in this problem refer to the Lebesgue measure. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Prove that, given $\varepsilon > 0$, there exists a smooth (i.e., infinitely differentiable) function g such that $\| f - g \|_p \leq \varepsilon$.

Question 38. All measure related statements in this problem refer to the Lebesgue measure.

Let $U \subset \mathbb{R}^n$ be an open and bounded domain, and $f : U \to \mathbb{R}$ a locally integrable function. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\varphi(x) = \begin{cases} a \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 0, \end{cases}$$

where

$$a^{-1} = \int_{B_1(0)} \exp\left(\frac{1}{|x|^2 - 1}\right) dx,$$

and $B_1(0)$ is the open ball of radius one centered at the origin.

For $\varepsilon > 0$, define $\varphi_\varepsilon(x)$ by

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon}).$$

(a) Prove that $f * \varphi_\varepsilon \in C^\infty(U_\varepsilon)$, where

$$U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\},$$

dist means distance, and $*$ is the convolution.

(b) Prove that $f * \varphi_\varepsilon$ converges to f in $L^p_{loc}(U)$ as $\varepsilon \to 0^+$.

Question 39. State and prove the open mapping theorem for Banach spaces.

Question 40. State and prove the inverse mapping theorem for Banach spaces.

Question 41. State and prove the closed graph theorem for Banach spaces.

Question 42. Let X be a Banach space and X' its dual.

(a) Define the weak topology on X and the weak-$*$ topology on X'.

(b) State and prove the Banach-Alaoglu theorem.

Question 43. Let X be a normed vector space and $\{x_n\}_{n=1}^\infty \subset X$ a sequence. Recall that the formal series $\sum_{n=1}^\infty x_n$ is called convergent if the sequence of the partial sums $\sum_{n=1}^N x_n$ converges in X as $N \to \infty$, and absolutely convergent if the sequence of the partial sums $\sum_{n=1}^N \| x_n \|$ converges in $\mathbb{R}$ as $N \to \infty$. Prove that X is a Banach space if and only if every absolutely convergent series is convergent.

Question 44. Let H be a Hilbert space. Denote its inner product by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\| \cdot \|$. We adopt the convention that $\langle \cdot, \cdot \rangle$ is linear in the second entry (and thus anti-linear in the first entry). Let $B : H \times H \to \mathbb{C}$ be a map satisfying: (i) $B(x, \alpha y + \beta z) = \alpha B(x, y) + \beta B(x, z)$, (ii) $B(\alpha x + \beta y, z) = \overline{\alpha} B(x, z) + \overline{\beta} B(y, z)$, and (iii) $|B(x, y)| \leq C \| x \| \| y \|$, for some constant C and all $x, y, z \in H$, $\alpha, \beta \in \mathbb{C}$, where $\overline{\alpha}$ is the complex conjugate of α. Prove that there exists a unique continuous linear map $A : H \to H$ such that

$$B(x, y) = \langle Ax, y \rangle,$$

for all $x, y \in H$.

Question 45. Let H be a Hilbert space. Let $\{x_n\}_{n=1}^{\infty} \subset H$ be a sequence that converges weakly to an element x. Prove that there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that the sequence of arithmetic means:

$$\{\frac{1}{k}\sum_{\ell=1}^{k} x_{n_k}\}_{k=1}^{\infty},$$

converges to x in H.