VANDERBILT UNIVERSITY, REAL ANALYSIS MIDTERM, SPRING 2016

Name:

**Directions.** Please read carefully the following directions:

- This exam contains three questions.
- Most problems require that you use results proven/stated in class/homework. When invoking such results, you do not have to prove them, unless the question itself is asking you to establish a result demonstrated in class or given as a homework. However, you do need to state clearly the theorems/definitions you are using.
- While there is not an absolute standard to decide which results you should establish in order to answer the questions versus those that you can quote from class/homework, you are expected to demonstrate mathematical knowledge of the subject, and provide proofs for the questions that you are being specifically asked.
- If the statement of a problem is not clear (for instance, you think there is a missing hypothesis, the question is ambiguous, the notation is confusing, etc), state clearly how you interpret it, and then solve it accordingly.
- A list of notations is provided at the end.

| **Question** | **Points** |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| **Total:** | |

**Question 1 [30 pts].** Let $X$ be a locally compact space and $\mu$ a measure on $X$.

a) State the definition of a $\mu$-measurable function.

b) Let $Y$ be a topological space. Prove that a function $f : X \to Y$ is $\mu$-measurable if and only if for every compact set $K \subseteq X$ there exist a $\mu$-negligible set $N \subseteq K$ and a partition of $K\backslash N$ into a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that $f|_{K_n}$ is continuous for each $n$.

c) Assume further that $X$ is $\sigma$-compact. Prove that a map $f : X \to Y$, $Y$ a topological space, is $\mu$-measurable if and only if there exists a partition of $X$ into a negligible set $N$ and a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that $f|_{K_n}$ is continuous for each $n$.

**Solution.**

a) Let $Y$ be a topological space and $f : X \to Y$. We say that $f$ is $\mu$-measurable if for every compact set $K \subseteq X$ and any $\varepsilon > 0$, there exists a compact set $K' \subseteq K$ such that $|\mu|(K\backslash K') \leq \varepsilon$ and $f|_{K'}$ is continuous.

a) Let $f$ be measurable and $K \subseteq X$ a compact set. Then $K$ is integrable (i.e., $\mu$-integrable), and by the characterization of integrable sets, given $\varepsilon_1$ we can find an integrable open set $U$ and a compact set $K_1'$, such that $K_1' \subseteq K \subseteq U$ and $|\mu|(U\backslash K_1') \leq \varepsilon_1$. Then

$$\begin{aligned}\varepsilon_1 &\geq |\mu|(U\backslash K_1') = |\mu|(U) - |\mu|(K_1')\\ &= |\mu|(U) - |\mu|(K) + |\mu|(K) - |\mu|(K_1') \geq |\mu|(K\backslash K_1') \geq 0.\end{aligned}$$

By assumption, we can find $K_1 \subseteq K_1'$ such that $|\mu|(K_1'\backslash K_1) \leq \varepsilon_1$ and $f|_{K_1}$ is continuous. Consider the set $K\backslash K_1$. It is integrable, thus mimicking the previous argument, given $\varepsilon_2$, we can find a compact set $K_2 \subseteq K\backslash K_1$ such that $f|_{K_2}$ is continuous and

$$|\mu|(K\backslash(K_1 \cup K_2))| = |\mu|((K\backslash K_1)\backslash K_2) \leq \varepsilon_2,$$

where we used that $(K\backslash K_1)\backslash K_2 = K\backslash(K_1 \cup K_2)$ since $K_1 \cap K_2 = \varnothing$. Continuing this process, we construct a sequence $\{K_n\}_{n=1}^{\infty}$ of pair-wise disjoint compact sets such that

$$|\mu|(K\backslash \bigcup_{n=1}^{k} K_n) \leq \frac{1}{k},$$

and $f|_{K_n}$ is continuous for each $n$. Setting $A_k = K\backslash \bigcup_{n=1}^{k} K_n$ and $N = \bigcap_{n=1}^{\infty} A_n$ we obtain the desired partition.

For the converse, notice that it follows from the stated condition that $|\mu|(K) = \sum_{n=1}^{\infty} |\mu|(K_n)$, so we can set $K' = \bigcup_{n=1}^{k} K_n$, choosing $k$ such that $|\mu|(K\backslash K') \leq \varepsilon$ for a given $\varepsilon > 0$.

c) Write $X = \bigcup_{n=1}^{\infty} K_n'$, where the $K_n'$ are compact and we can assume the sequence to be increasing. Let $L_1 = K_1$ and $L_n = K_n\backslash K_{n-1}$ for $n \geq 2$, so that the sets $L_n$ are pair-wise disjoint and $\bigcup_{n=1}^{\infty} L_n = X$. Because each $L_n$ is integrable, it can be written as

$$L_n = N_n \cup \bigcup_{m=1}^{\infty} K_{nm},$$

where $N_n$ is negligible and the $K_{nm}$'s are compact. Because $f$ is measurable, by part b) each $K_{nm}$ can be partitioned as

$$K_{nm} = N_{nm} \cup \bigcup_{k=1}^{\infty} K_{nmk},$$

where $N_{nm}$ is negligible and $f|_{K_{nmk}}$ is continuous, which implies the result.

**Question 2 [30 pts].** Let $X$ be a locally compact $\sigma$-compact space with a measure $\mu$ on it.

a) State the definition of the essential supremum of a function.

b) Prove that $\mathscr{L}^\infty(X)$ is complete.

**Solution.**

a) For any measurable function,

$$M_\infty(f) = \inf\{\alpha \in \mathbb{R} \,|\, f(x) \leq \alpha \text{ almost everywhere.}\}.$$

b) Since the topology on $\mathscr{L}^\infty(X)$ is generated by a single semi-norm, it suffices to consider Cauchy sequences. Let $\{f_n\}_{n=1}^\infty \subset \mathscr{L}^\infty(X)$ be a Cauchy sequence. Given $k \in \mathbb{N}$, we can find a $N_k$ such that $N_\infty(f_m - f_n) \leq \frac{1}{k}$ for all $m, n \geq N_k$. For each $m, n \geq N_k$, set $A_{mnk} = \{x \in X \,|\, |f_m(x) - f_n(x)| > \frac{1}{k}\}$. Then $A_{mnk}$ is negligible, and thus is their union $A$. It follows that $\{f_n(x)\}_{n=1}^\infty$ converges uniformly on $X \backslash A$; set $f(x)$ to be its limit (defined almost everywhere). $f$ is then bounded on $X \backslash A$ and by Egoroff's theorem it is measurable; hence $f \in \mathscr{L}^\infty(X)$. Because $\{f_n\}_{n=1}^\infty$ converges uniformly to $f$ on the complement on of a negligible set, we conclude (from the characterization of convergence in $\mathscr{L}^\infty(X)$) that $\{f_n\}_{n=1}^\infty$ converges to $f$ in $\mathscr{L}^\infty(X)$. (Alternatively, we can use that $N_\infty(f - f_n) \leq \frac{1}{k}$ for $n \geq N_k$.)

**Question 3 [40 pts].** Let $X$ be a locally compact $\sigma$-compact space. Let $\mu$, $\lambda$, and $\nu$ be positive measures on $X$. Suppose that every $\mu$-measurable set is $\lambda$-measurable and $\nu$-measurable, that every $\mu$-negligible set is $\lambda$-negligible and $\nu$-negligible, and that $\lambda(X) = 1 = \nu(X)$.

a) Let $\Sigma$ be the collection of all $\mu$-measurable sets. Explain why the quantity

$$\sup_{A\in\Sigma} |\lambda(A) - \nu(A)|$$

is a well-defined real number.

b) Prove there there exist $\mu$-integrable functions $f$ and $g$, whose equivalence classes are uniquely determined by $\lambda$ and $\nu$, respectively, such that

$$\sup_{A\in\Sigma} |\lambda(A) - \nu(A)| = \frac{1}{2}\int |f - g|\, d\mu.$$

**Solution.**

a) Since $0 \leq \lambda(A) \leq 1$ and $0 \leq \nu(A) \leq 1$ for any $A \in \Sigma$, we have $0 \leq |\lambda(A) - \nu(A)| \leq 2$, thus the result.

b) Since every $\mu$-negligible set is also $\lambda$-negligible, the Radon-Nikodym derivative $f = \frac{d\lambda}{d\mu}$ exists. $f$ is a locally $\mu$-integrable function whose equivalence class is uniquely determined. Since

$$1 = \lambda(X) = \int d\lambda = \int \frac{d\lambda}{d\mu}\, d\mu = \int f\, d\mu,$$

we see that $f$ is $\mu$-integrable. Similarly for $g = \frac{d\nu}{d\mu}$. For any $A \in \Sigma$, it holds that

$$\lambda(A) = \int_A f\, d\mu,$$

and

$$\nu(A) = \int_A g\, d\mu.$$

For any $A \in \Sigma$, we have

$$\begin{aligned} 0 = \lambda(X) - \nu(X) &= \int (f - g)\, d\mu \\ &= \int_A (f - g)\, d\mu + \int_{A^c} (f - g)\, d\mu, \end{aligned}$$

thus

$$\int_A (f - g)\, d\mu = -\int_{A^c} (f - g)\, d\mu.$$

Therefore,

$$\begin{aligned} 2\left|\int_A (f - g)\, d\mu\right| &= \left|\int_A (f - g)\, d\mu\right| + \left|\int_A (f - g)\, d\mu\right| \\ &= \left|\int_A (f - g)\, d\mu\right| + \left|\int_{A^c} (f - g)\, d\mu\right| \\ &\leq \int_A |f - g|\, d\mu + \int_{A^c} |f - g|\, d\mu \\ &= \int |f - g|\, d\mu. \end{aligned}$$

Hence,

$$|\lambda(A) - \nu(A)| = \left| \int_A f d\mu - \int_A g \, d\mu \right| \leq \frac{1}{2} \int |f - g| \, d\mu.$$

Since $A \in \Sigma$ is arbitrary,

$$\sup_{A \in \Sigma} |\lambda(A) - \nu(A)| \leq \frac{1}{2} \int |f - g| \, d\mu.$$

Next, let $A_+ = \{x \in X \mid f(x) - g(x) > 0\}$, $A_- = \{x \in X \mid f(x) - g(x) < 0\}$, and $A_0 = \{x \in X \mid f(x) - g(x) = 0\}$. Notice that $A_+$, $A_-$, and $A_0$ are $\mu$-measurable. Compute

$$\begin{aligned}
\int |f - g| \, d\mu &= \int_{A_+} |f - g| \, d\mu + \int_{A_-} |f - g| \, d\mu \\
&= \int_{A_+} (f - g) \, d\mu - \int_{A_-} (f - g) \, d\mu \\
&= \lambda(A_+) - \nu(A_+) - (\lambda(A_-) - \nu(A_-)) \\
&\leq |\lambda(A_+) - \nu(A_+)| + |\lambda(A_-) - \nu(A_-)| \\
&\leq 2 \sup_{A \in \Sigma} |\lambda(A) - \nu(A)|,
\end{aligned}$$

finishing the proof.