Here is the proof of the following theorem discussed in class (notation the same as in class):

Theorem 1. *Let V be a non-empty set of functions in $\mathscr{I}_+(X;\mathbb{R})$, directed for the relation $\leq$, and μ a positive measure on X. Then*

$$\mu^*(\sup_{g\in V}) = \sup_{g\in V} \mu^*(g).$$

Proof. Set $f = \sup_{g\in V} g$. Since any function in $g \in \mathscr{I}_+(X;\mathbb{R})$ is the upper envelope of the functions in $\mathscr{K}_+(X;\mathbb{R})$ that are $\leq$ than g, we have $\mu^*(g) \leq \mu^*(f)$, $g \in V$. Thus it suffices to show that $\mu(\psi) \leq \sup_{g\in V} \mu^*(g)$ for every $\psi \in \mathscr{K}_+(X;\mathbb{R})$ such that $\psi \leq f$.

For any $g \in V$, let Φ_g be the collection of functions $\varphi \in \mathscr{K}_+(X;\mathbb{R})$ such that $\varphi \leq g$. Let $\Phi = \cup_{g\in V} \Phi_g$. Thus $f = \sup_{\varphi\in\Phi} \varphi$. Let $\psi \in \mathscr{K}_+(X;\mathbb{R})$ be such that $\psi \leq f$. ψ is the upper envelope of the functions $\varphi \in \Phi$ such that $\varphi \leq \psi$, i.e.,

$$\psi = \sup_{\varphi\in\Phi} \inf(\varphi, \psi). \tag{1}$$

Notice that $\text{supp}(\inf(\varphi, \psi)) \subseteq \text{supp}(\psi) \subseteq K$ for some compact K. Recall (i) that the topology of uniform converge and that of convergence in compact sets agree on compact subsets of X; and (ii) that in class we proved that if the upper envelope of a family of functions V' in $\mathscr{C}(K;\mathbb{R})$ is finite and continuous, then such an upper envelope can be uniformly approximated by functions in V'. (i), (ii) and (1) thus give

$$\lim_{\varphi\in\Phi} (\inf(\psi, \varphi)) = \psi. \tag{2}$$

Continuity of μ now implies $\sup_{\varphi\in\Phi} \mu(\inf(\psi, \varphi)) = \mu(\psi)$. On the other hand, since each $\varphi \in \Phi$ belongs to Φ_g for some g,

$$\mu(\inf(\psi, \varphi)) \leq \mu(\varphi) \leq \mu^*(g) \leq \sup_{g\in V} \mu^*(g).$$

□