REAL ANALYSIS, HW 8

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\,d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Question 1. Consider $\mathscr{K}(X)$ with its original topology (i.e., the inductive limit of locally convex topologies) and denote by $\mathscr{K}^p(X)$ the same set with topology induced from $\mathscr{F}^p(X)$. Discuss the relation between these two topologies, including questions of coarseness, Hausdorff properties, and completeness.

Question 2. Show that any $f \geq 0$ that belongs to $\mathscr{L}^p(X)$ is the limit, in the $\mathscr{L}^p(X)$ topology, of a sequence of functions in $\mathscr{K}_+(X;\mathbb{R})$.

Question 3. Let f be a numerical function such that $|f|^p$ is integrable (i.e., $|f|^p \in \mathscr{L}^1(X)$). Show that it does not necessary follow that $f \in \mathscr{L}^p(X)$.

Question 4. Prove the integration term-by-term theorem stated in class: if $\{f_n\}_{n=1}^\infty$ is a sequence of integrable functions such that the series of the f_n's converges a.e., and if there exists a $g \geq 0$ such that $|\sum_{k=1}^n f_k(x)| \leq g(x)$ a.e. and for all n, then the series of the f_n's is integrable and

$$\int \sum_{n=1}^\infty f_n \, d\mu = \sum_{n=1}^\infty \int f_n \, d\mu.$$

Question 5. Consider the sequence of functions in $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} x^n & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Does $\{f_n\}_{n=1}^\infty$ converge to a limit in the L^p topology? If yes, exhibit the limit. (The measure is the Lebesgue measure).

Question 6. Recall that we say that a set $A \subseteq X$ is integrable if its characteristic function χ_A is integrable, in which case its measure $\mu(A)$ is defined as $\int \chi_A \, d\mu$. Prove the following statements.

1) If $\{A_n\}_{n=1}^N$ is a finite family of integrable sets, then $\cup_{n=1}^N A_n$ is integrable and

$$|\mu|(\bigcup_{n=1}^N A_n) \leq \sum_{n=1}^N |\mu|(A_n),$$

with equality in the case where the sets are pairwise disjoint.

2) If A and B are integrable and $B \subset A$, then $A \backslash B$ is integrable and $\mu(A\backslash B) = \mu(A) - \mu(B)$.

3) The intersection of a countable family of integrable sets is integrable.

4) If $\{A_n\}_{n=1}^\infty$ is a decreasing sequence of integrable sets, then

$$\mu(\bigcap_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n).$$

5) Let $\{A_n\}_{n=1}^\infty$ be an increasing sequence of integrable sets. Then $\cup_{n=1}^\infty A_n$ is integrable if and only if $\sup_n |\mu|(A_n) < \infty$, and in this case

$$\mu(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(A_n).$$

If $\sum_{n=1}^{\infty} |\mu|(A_n) < \infty$, then $\cup_{n=1}^{\infty} A_n$ is integrable and

$$|\mu|(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} |\mu|(A_n),$$

with equality in the case where the sets are pairwise disjoint.

Question 7. Let $\mathcal{G}$ be a family of integrable closed sets directed for $\supseteq$. Then

$$A = \bigcap_{G \in \mathcal{G}} G$$

is integrable,

$$|\mu|(A) = \inf_{G \in \mathcal{G}} |\mu|(G),$$

and

$$\mu(A) = \lim_{G \in \mathcal{G}} \mu(G).$$

Question 8. Show that a set A is integrable if and only if for every $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $|\mu|^*(A \backslash K) \leq \varepsilon$. In this case, the measure $|\mu|(A)$ is the supremum of $|\mu|(K)$ for K compact and contained in A.

Question 9. Show that for any integrable set A, there exist (i) a set B that is a countable intersection of integrable open sets, such that $A \subseteq B$ and $B \backslash A$ is negligible; (ii) a set C that is a countable union of pairwise disjoint compact sets, such that $C \subseteq A$ and $A \backslash C$ is negligible.

Question 10. Show that for any open set U, $|\mu|^*(U)$ is the supremum of $|\mu|(K)$ for K compact and contained in U. Notice that it is not assumed that $|\mu|^*(U) < \infty$.