REAL ANALYSIS, HW 7

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\,d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Question 1. Prove that any subset of a set of zero measure has zero measure, and that a countable union of zero measure sets has zero measure.

Question 2. Prove that a lower semi-continuous function $f \geq 0$ is negligible if and only if f is zero on the support of μ. *Hint:* First prove that if $f \in \mathscr{K}(X;\mathbb{C})$ vanishes on $\operatorname{supp}(\mu)$, then $\mu(f) = 0$. Next, show that if μ is a positive measure on X and $f \in \mathscr{K}_+(X;\mathbb{R})$ is such that $\mu(f) = 0$, then f vanishes on $\operatorname{supp}(\mu)$. Use these two statements to conclude the result.

Question 3. Let $f \geq 0$ be a numerical function defined on X. Prove that if $|\mu|^*(f) < \infty$, then f is finite a.e.

Question 4. Let $f \geq$ and $g \geq 0$ be numerical functions defined on X. Prove that if $f(x) = g(x)$ a.e., then $|\mu|^*(f) = |\mu|^*(g)$.

Question 5. Establish the statements below. All functions are assumed to be numerical functions on X defined a.e.

(a) $f \sim g$ if and only if $f(x) = g(x)$ a.e. on $\operatorname{supp}(\mu)$.

(b) The sum and product of functions finite a.e. are finite a.e.

(c) The ordering relation $\widetilde{f} \leq \widetilde{g}$ between equivalence classes as stated in class is well-defined.

(d) If $\{\widetilde{f}_n\}_{n=1}^{\infty}$ is a sequence of equivalence classes of functions with values in $\overline{\mathbb{R}}$, then $\sup_n \widetilde{f}_n$, $\inf_n \widetilde{f}_n$, $\limsup_{n\to\infty} \widetilde{f}_n$ and $\liminf_{n\to\infty} \widetilde{f}_n$ are well-defined.

Question 6. Show that the complement of the support of μ is the largest negligible open set in X. Use this result to show that if f and g are continuous functions from X to $\mathbb{C}$, then they are equivalent if and only if $f(x) = g(x)$ for every $x \in \operatorname{supp}(\mu)$.

Question 7. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathscr{F}^p(X)$ such that $\sum_{n=1}^{\infty} N_p(f) < \infty$. Show that the series $\sum_{n=1}^{\infty} f_n$ converges absolutely a.e. Define f to be equal to the value of this series at points x where it converges and zero otherwise. Prove that $f \in \mathscr{F}^p(X)$, and that

$$N_p(f - \sum_{k=1}^{n} f_k) \leq \sum_{k=n+1}^{\infty} N_p(f_k).$$

Conclude that f is the sum of the series $\sum_{n=1}^{\infty} f_n$ in $\mathscr{F}^p(X)$.

Question 8. Let μ and ν be positive measures on X and $\lambda > 0$ a number. Show that $(\lambda\mu)^* = \lambda\mu^*$ and $(\mu + \nu)^* = \mu^* + \nu^*$. Show also that if $\mu \leq \nu$ then $\mu^* \leq \nu^*$.

Question 9. Give examples that show the following statements to be true: (i) A Cauchy sequence in $\mathscr{L}^p(X)$ may not converge at any point of X. (ii) If $f \in \mathscr{L}^p(X)$, it is not always the case that there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{K}(X)$ such that $\{f_n(x)\}_{n=1}^{\infty}$ converges everywhere to a function that is almost everywhere equal to $f(x)$. (iii) If $f_n(x) \to f(x)$ a.e., it does not follow that f_n converges to f in $\mathscr{L}^p(X)$.

Question 10. Show that $L^p(X)$ with an order relation induced from $\mathscr{F}^p(X)$ is a Riesz space.