# REAL ANALYSIS, HW 6

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The notation used below follows the one used in class and should be self-explanatory. Directions similar to those of previous homework assignments continue to hold, in particular (i) sometimes a definition that has not been given in class is used in an exercises. It is expected that students will be able to figure out the obvious interpretation, but you can always consult the literature if necessary; (ii) problems are written in an understandable, but loose fashion. When necessary or convenient, first make a precise statement of what is being asked before presenting your solution.

Unless stated otherwise, the following notation is adopted throughout (which is the same used in class):

| | |
|---|---|
| supp | Support of a function or a measure |
| $X$ | Locally compact (topological) space |
| $K$ | Compact set in $X$ |
| $E$ | Locally convex (topological vector) space |
| $\mathscr{C}(X;E)$ | Space of continuous functions from $X$ to $E$ endowed with the uniform topology |
| $\mathscr{C}_{c.o.}(X;E)$ | Space of continuous functions from $X$ to $E$ endowed with the compact-open topology |
| $\mathscr{C}_c(X;E)$ | Space of continuous functions from $X$ to $E$ with compact support endowed with the compact-open topology |
| $\mathscr{C}(K;E)$ | Space of continuous functions from $K$ to $E$ endowed with the topology inherited from $\mathscr{C}(X,E)$ |
| $\mathscr{K}(X;E)$ | Space of continuous functions from $X$ to $E$ with compact support endowed with the inductive limit of locally convex topologies |
| $\mathscr{K}(X,A;E)$ | Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq A$ |
| $\mathscr{K}(X,K;E)$ | Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq K$ endowed with the topology of compact convergence |
| $\mathscr{K}_+(X;\mathbb{R})$ | Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$ |
| $\mathscr{M}(X;\mathbb{C})$ | Space of measures on $X$ |
| $\mathscr{M}(X;\mathbb{R})$ | Space of real measures on $X$ |
| $\mathscr{M}_+(X;\mathbb{R})$ | Space of positive measures on $X$ |
| $\mathscr{I}_+(X;\mathbb{R})$ | Space of positive (non-negative) lower semi-continuous functions on $X$ |
| $\mu^*(f)$ | Upper integral of $f$ (with respect to the positive measure $\mu$), also denoted $\int^* f\, d\mu$ |
| $\chi_A$ | Characteristic function of the set $A$ |
| $\mu^*(A)$ | Outer measure of $A$ (with respect to the positive measure $\mu$) |

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering on $\mathscr{K}(X;\mathbb{R})$ and $\mathscr{M}(X;\mathbb{R})$ is as defined in class and denoted $\leq$.

**Question 1.** Discuss the relations between the spaces $\mathscr{C}(X;E)$, $\mathscr{C}_{c.o.}(X;E)$, $\mathscr{C}_c(X;E)$, and $\mathscr{K}(X;E)$, providing proofs of your statements. For instance, which is finer/coarser than each? Do they induce the same topology on $\mathscr{K}(X,K;E)$? How do you state convergence of nets in each of these spaces? How would your statements change/be refined if $X$ is compact? Idem if the topology of $E$ is given by a norm.

**Question 2.** Let $X$ be compact. Let $V$ be a set in $\mathscr{C}(X;\mathbb{R})$ which is directed for the relation $\leq$. Suppose that for any $f_2, f_2 \in V$, there exists $f_3 \in V$ such that $f_1 \leq f_3$ and $f_2 \leq f_3$. Assume that the upper envelope $f$ of $V$ is finite and continuous. Prove that $f$ can be uniformly approximated by functions in $V$.

**Question 3.** Assume the same hypotheses of question 2 and suppose that $V$ is given by $V = \{f_\alpha\}_{\alpha\in A}$. Then there exists a $A' \subseteq A$ such that

$$\lim_{\alpha\in A'} \parallel f_\alpha(x) - f(x) \parallel = 0,$$

where as usual

$$\parallel g \parallel = \sup_{x\in X} |g(x)|.$$

**Question 4.** Show that $f$ is lower semi-continuous if and only if $-f$ (minus $f$) is upper semi-continuous.

**Question 5.** Prove that $\mu^*$ is increasing on $\mathscr{I}_+(X;\mathbb{R})$, $\mu^*(\lambda f) = \lambda\mu^*(f)$, $\lambda > 0$, and that $\mu^*(f + g) = \mu^*(f) + \mu^*(g)$, $f, g \in \mathscr{I}_+(X;\mathbb{R})$.

**Question 6.** State the definition of the Lebesgue measure. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (\mathbb{R}\backslash\mathbb{Q}) \cap [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Show that

$$\int^* f\, d\mu = 1,$$

where $\mu$ is the Lebesgue measure. Conclude that the outer measure of the set of irrational numbers between 0 and 1 is equal to one.

**Question 7.** In this problem you will be asked to prove some properties of the *Cantor set,* defined as follows.

Let $C_0 = [0,1]$. Let $C_1$ be the set obtaining by removing the open middle-third interval of $C_0$, i.e.,

$$C_1 = C_0\backslash(\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

Define $C_n$ inductively by removing the open middle-third of each interval of $C_{n-1}$. For instance,

$$\begin{aligned} C_2 &= \left([0, \frac{1}{3}]\backslash(\frac{1}{9}, \frac{2}{9})\right) \cup \left([\frac{2}{3}, 1]\backslash(\frac{7}{9}, \frac{8}{9})\right) \\ &= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]. \end{aligned}$$

The Cantor set $C$ ($C$ for Cantor) is defined as the set of points in $[0,1]$ that belong to $C_n$ for every $n$.

(a) Show that the Cantor set is not empty and contains no open interval.

(b) Show that

$$C = [0,1] \backslash \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{3^{i-1}-1} (\frac{3j+1}{3^i}, \frac{3j+2}{3^i}).$$

(c) Show that

$$C = \bigcap_{i=1}^{\infty} \bigcap_{j=0}^{3^{i-1}-1} \left( [0, \frac{3j+1}{3^i}] \cup [\frac{3j+2}{3^i}, 1] \right).$$

(d) Show that

$$\int^* \chi_C \, d\mu = 0,$$

where $\mu$ is the Lebesgue measure. Conclude that $\mu^*(C) = 0$.

**Question 8.** This problem shows that the construction of the Cantor set can be modified so that $\mu^*(C) > 0$. In such cases $C$ is sometimes referred to as a *fat Cantor set.* The ensuing construction is modeled after that of exercise 7, thus make sure you understand problem 7 first.

Fix a number $z$ such that $0 < z < \frac{1}{3}$, and let $F_0 = [0,1]$. Let $F_1$ be the set obtained from $F_0$ by removing an open interval of length $z$ from the middle of $F_0$, i.e., $F_1 = F_0 \backslash U_{1,1}$, where $U_{1,1} = (\frac{1}{2} - \frac{z}{2}, \frac{1}{2} + \frac{z}{2})$. Notice that $F_1$ is the union of two closed intervals. Define $F_2$ by removing from $F_1$ two open intervals $U_{2,1}$ and $U_{2,2}$ of length $z^2$ each, with $U_{1,1}$ removed from the middle of the first closed interval that forms $F_1$ and $U_{2,2}$ removed from the middle of the second closed interval that forms $F_1$. Thus $F_2$ is the union of four closed intervals. Define $F_{n+1}$ inductively by removing $2^n$ open intervals $U_{n+1,1}, \dots, U_{n+1,2^n}$ of length $z^{n+1}$ from the middle of the $2^n$ closed intervals that form $F_n$.

Set

$$F = \bigcap_{n=0}^{\infty} F_n.$$

($F$ for fat).

(a) Show that $F$ is not empty and contains no open interval.

(b) Let $G = [0,1] \backslash F$. Show that

$$\int^* \chi_G \, d\mu < 1,$$

where $\mu$ is the Lebesgue measure. Conclude that $\mu^*(G) < 1$ and that $\mu^*(F) > 0$.

(c) Obviously, the above construction depends on the number $z$ we have chosen. Prove that given $\varepsilon > 0$, there exists a $z \in (0, \frac{1}{3})$ such that $\mu^*(F) > 1 - \varepsilon$. I.e., we can make $\mu^*(F)$ as close to one as we want.

**Question 9.** Let $N$ be a countable subset of $\mathbb{R}$. Prove that $\mu^*(N) = 0$, where $\mu$ is the Lebesgue measure. In particular, $\mu^*(\mathbb{Q}) = 0$.

**Question 10.** (not really a question, just something for you to think about) As discussed in class, the (outer) Lebesgue measure of a set is a generalization of the usual notion of volume or length. Let us thus agree to call the Lebesgue outer measure of a set its volume. Problem 9 shows that the volume of a countable set is zero, and problem 6 that the volume of the set of irrational numbers between zero and one is equal to one. We might be tempted to think then that uncountable sets have positive volume. To see that this is not necessarily the case, convince yourself that the Cantor set

$C$ is uncountable, although its volume is zero as showed in question 7. Therefore, uncountability is not sufficient to guarantee positive volume. The volume of the fat Cantor set (which is also uncountable) on the other hand, is positive, as showed in question 8. Reflect on the fact that the fat Cantor set is an uncountable closed set that contains no open interval but whose volume is as close to one as we want. Reflect also on the fact that $\mathbb{Q}$ is a dense subset of the real line whose volume is zero. What do these things tell you about measure theory?