REAL ANALYSIS, HW 5

VANDERBILT UNIVERSITY

The notation used below follows the one used in class and should be self-explanatory. Directions similar to those of previous homework assignments continue to hold, in particular (i) sometimes a definition that has not been given in class is used in an exercises. It is expected that students will be able to figure out the obvious interpretation, but you can always consult the literature if necessary; (ii) problems are written in an understandable, but loose fashion. When necessary or convenient, first make a precise statement of what is being asked before presenting your solution.

Unless stated otherwise, the following notation is adopted throughout (which is the same used in class):

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq A$
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering on $\mathscr{K}(X;\mathbb{R})$ and $\mathscr{M}(X;\mathbb{R})$ is as defined in class and denoted $\leq$.

Question 1. In this problem, you are asked to prove several natural properties of measures.

Show that the map

$$(f_1, f_2) \in \mathscr{K}(X;\mathbb{R}) \times \mathscr{K}(X;\mathbb{R}) \mapsto f_1 + i f_2 \in \mathscr{K}(X;\mathbb{C})$$

is a topological isomorphism. Let $\mu \in \mathscr{M}(X;\mathbb{C})$ and μ_0 be its restriction to $\mathscr{K}(X;\mathbb{R})$. Show that $\mu_0 \in \mathscr{M}(X;\mathbb{R})$, and that every measure on X can in fact be identified with its restriction to $\mathscr{K}(X;\mathbb{R})$.

Let $\mu \in \mathscr{M}(X;\mathbb{C})$. Define its **complex conjugate** $\overline{\mu}$ by

$$\overline{\mu}(f) = \overline{\mu(\overline{f})}.$$

Show that $\overline{\mu}$ is a measure. Next, define a measure to be real if $\overline{\mu} = \mu$. Show that this definition is equivalent to the definition of real measures given in class.

Finally, show that the measures $Re(\mu)$ and $Im(\mu)$ defined by $(\mu+\overline{\mu})/2$ and $(\mu-\overline{\mu})/2i$, respectively, are real. They are called the **real** and **imaginary parts** of μ, respectively.

Question 2. Prove that every positive linear form on $\mathscr{K}(X;\mathbb{R})$ is a positive measure.

Question 3. Let $\mu \in \mathscr{M}(X;\mathbb{C})$. Show that for every $f \in \mathscr{K}_+(X;\mathbb{R})$, the non-negative number

$$L(f) = \sup_{\substack{g\in\mathscr{K}(X;\mathbb{C})\\ |g|\leq f}} \mu(g)$$

is finite. Show that if $f_1, f_2 \in \mathscr{K}_+(X;\mathbb{R})$, then $L(f_1 + f_2) = L(f_1) + L(f_2)$ (recall that we did something similar in class). Conclude, by an argument similar to the one used in class, that L can be extended uniquely to a positive measure on X. Denote this measure by $|\mu|$. It is called the **absolute value** of μ. Show that

$$\left|\int f\, d\mu\right| \leq \int |f|\, d|\mu|,$$

for every $f \in \mathscr{K}(X;\mathbb{C})$. Show also that $|\mu + \nu| \leq |\mu| + |\nu|$ and that $|\overline{\mu}| = |\mu|$, where μ and ν are measures on X.

Question 4. Recall that in class we proved that $\mathscr{M}(X;\mathbb{R})$ is a lattice (= an ordered set such that every two elements have a supremum and infimum). We showed this by proving first that $\sup(\mu, 0)$ is well-defined for any $\mu \in \mathscr{M}(X;\mathbb{R})$, and that this then implied that $\sup(\mu,\nu)$ and $\inf(\mu,\nu)$ are well-defined for any two real measures μ and ν. The proof done in class is a particular case of the following more general result: let E be an ordered vector space such that $\sup(x, 0)$ exists for every $x \in E$. Then E is a Riesz sapce (= an ordered vector space that is also a lattice). The proof of this statement is identical to the one given in class because, once we had showed that $\sup(\mu, 0)$ always exists, the remaining of the proof relied only on the order structure of $\mathscr{M}(X;\mathbb{R})$. In this exercise you are asked to prove the following alternative characterization of a Riesz space.

Let E be an ordered vector space, and E^+ the subset of $x \in E$ such that $x \geq 0$. Then, E is a Riesz space if and only if the following happens (i) E^+ generates E, i.e., every $x \in E$ is of the form $x = x_1 - x_2$, where x_1 and x_2 belong to E^+; and (ii) $\sup(x, y)$ always exists for every pair of elements $x, y \in E^+$.

Question 5. Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of X. Suppose that for each $\alpha \in A$, we are given a measure μ_α on U_α. Assume that for each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the restrictions of μ_α and μ_β to $U_\alpha \cap U_\beta$ agree. Prove that there exists a unique measure μ on X such that $\mu|_{U_\alpha} = \mu_\alpha$ for each $\alpha \in A$.

Question 6. Show that a measure is zero if and only if its support is empty. Let $\mu \in \mathscr{M}(X;\mathbb{C})$. Show that $\operatorname{supp}(\mu) = \operatorname{supp}(|\mu|)$. If μ is a real measure, show that its support is the union of the supports of μ^+ and μ^-.

Question 7. Let $\mu \in \mathscr{M}(X;\mathbb{C})$. Prove that μ is a discrete measure (defined in class with a family β) if and only if $\operatorname{supp}(\mu)$ is a discrete closed subset of X.

Question 8. Let $\{x_i\}_{i=1}^n$ be a set of distinct points in X, and μ a measure whose support is contained in $\cup_{i=1}^n \{x_i\}$. Prove that μ is a linear combination of the measures δ_{x_i}, where δ_x is the Dirac measure at x.

Question 9. Let X and Y be locally compact spaces, and λ and μ measures on X and Y, respectively. Let $K \subseteq X$ and $L \subseteq Y$ be compact sets. Prove that for every function

$$f \in \mathscr{K}(X \times Y, K \times L;\mathbb{C}),$$

the function

$$y \mapsto h(y) = \int f(x,y)\, d\lambda(x),$$

belongs to $\mathscr{K}(Y, L; \mathbb{C})$. Use this then to prove the existence and uniqueness of the product measure $\lambda \otimes \mu$ stated in class.

Question 10. Let $\lambda \in \mathscr{M}(X;\mathbb{C})$ and $\mu \in \mathscr{K}(Y;\mathbb{C})$. Show that $|\lambda \otimes \mu| = |\lambda| \otimes |\mu|$. If λ and μ are real measures, then show that

$$(\lambda \otimes \mu)^+ = \lambda^+ \otimes \mu^+ + \lambda^- \otimes \mu^-,$$

and

$$(\lambda \otimes \mu)^- = \lambda^+ \otimes \mu^- + \lambda^- \otimes \mu^+.$$