REAL ANALYSIS, HW 3

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The notation used below follows the one used in class and should be self-explanatory. Directions similar to those of previous homework assignments continue to hold, in particular (i) sometimes a definition that has not been given in class is used in an exercises. It is expected that students will be able to figure out the obvious interpretation, but you can always consult the literature if necessary; (ii) problems are written in an understandable, but loose fashion. When necessary or convenient, first make a precise statement of what is being asked before presenting your solution.

Question 1. Let (X, d) be a metric space and $(E, |\cdot|)$ a normed vector space. Let $f : X \to E$. Prove that f is continuous if and only if the following holds. For any sequence $\{x_i\} \subset X$ that converges to a point $x \in X$, i.e., $\lim_{i\to\infty} d(x_i, x) = 0$, the sequence $\{f(x_i)\}$ in E converges to $f(x)$, i.e., $\lim_{i\to\infty} |f(x_i) - f(x)| = 0$.

Question 2. Recall that in class we proved the following. Let E be a topological vector space and x' a linear form on E that is not identically zero. Then x' is continuous if and only if the hyperplane defined by $\langle x', x\rangle = 0$ is closed (where by $\langle x', x\rangle = 0$ we mean "the set of $x \in E$ such that..."). Recall that the proof involved a step (called in class "fact 2") that used that the co-dimension of $\overline{V}$ is equal to one (notation as in class). Show that this is indeed the case. In fact, show the following more general statement, which implies the desired properties of $\overline{V}$. Suppose L is a proper subspace of E. Then L does not contain any open set of E. i.e., if $U \subset E$ is open, then $U \not\subset L$. Consider next a vector space with the discrete topology. Does it not provide a counter-example to what you just proved?

Question 3. Let E be a topological vector space and V a subspace. Consider the quotient (in the linear algebra sense) space $E' = E/V$, and topologize it as done in class. Prove that E' is Hausdoff if and only if V is closed.

Question 4. Let V be a subspace of a topological vector space E. Give a sufficient condition for V to be dense in E. If such a sufficient condition is met, can you say something about the dimension of E?

Question 5. In our study of topological vector spaces in class, our proofs made explicit use of the continuity of multiplication by scalars. Did we use the continuity of addition at all?

Question 6. (This problems deals with the more general definition of locally convex spaces mentioned in class). Recall that if $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X, we say that $\mathcal{T}$ is coarser than $\mathcal{T}'$, and write $\mathcal{T} \subset \mathcal{T}'$, if every open set of $\mathcal{T}$ is also an open set of $\mathcal{T}'$ (so $\mathcal{T}$ has "fewer" open sets than $\mathcal{T}'$). We also say that $\mathcal{T}$ is weaker[1] than $\mathcal{T}'$. If we consider a property P related to the set X, we can define the *weakest* or *coarsest* topology on X such that P holds.

[1] The terminology weaker has the following justification. We can define the concept of convergence of a sequence in a general topological space (don't you know that? Check a topology book or, better, try to figure out yourself what the natural definition would be. For this, think in terms of sequences and open balls in $\mathbb{R}^n$, and then try replacing open balls by open sets). Then, more sequences converge with respect to $\mathcal{T}$ than $\mathcal{T}'$. I.e., the requirements for a sequence to converge with respect to $\mathcal{T}'$ are stronger.

Let E be a vector space. Let $\{p_\alpha\}_{\alpha\in A}$ be a family of maps $p_\alpha : E \to [0,\infty)$ such that

(i) $p_\alpha(x+y) \leq p_\alpha(x) + p_\alpha(y)$,

and

(ii) $p_\alpha(\lambda x) = |\lambda||p_\alpha(x)$,

for all $x, y \in E$, $\alpha \in A$, and $\lambda \in \mathbb{R}$. A map satisfying (i) and (ii) is called a **semi-norm**. We shall study semi-norms in detail later on. Together with the family $\{p_\alpha\}_{\alpha\in A}$, E is called a **locally convex space**.

(a) Let $\mathcal{T}$ be the weakest topology on E such that all the functions p_α and the vector space operation of addition is continuous[2]. Prove that a local base at the origin of E is given by the sets $\{N_{\alpha_1\,\alpha_2\,\cdots\,\alpha_n\,\varepsilon} \,|\, \alpha_1, \alpha_2 \dots, \alpha_n \in A,\, \varepsilon > 0\}$ where

$$N_{\alpha_1\,\alpha_2\,\cdots\,\alpha_n\,\varepsilon} = \{x \in E \,|\, p_{\alpha_i}(x) < \varepsilon,\, i = 1, \dots, n\}.$$

(b) Prove that $(E, \mathcal{T})$ is a topological vector space.

(c) The family $\{p_\alpha\}_{\alpha\in A}$ is said to separate points if

$$p_\alpha(x) = 0 \text{ for all } \alpha \in A \implies x = 0.$$

Show that if $\{p_\alpha\}_{\alpha\in A}$ separates points, then $(E, \mathcal{T})$ is Hausdorff. Conclude then that if we define a locally convex space as above and assume the separating points property, then we obtain a locally convex topological vector space in the sense of the definition given in class.

(d) Assume that the indexing set A is countable. Set

$$d(x,y) = \sum_{i=1}^\infty \frac{1}{2^i} \frac{p_i(x-y)}{1+p_i(x-y)}.$$

Prove that d is well-defined and gives a metric on E. Furthermore, the topology generated by d is equivalent to $\mathcal{T}$.

(e) Let Ω be an open set in $\mathbb{R}^n$, and $C^\infty(\Omega)$ be the space of all real-valued smooth (i.e., infinitely many times differentiable) functions defined in Ω that are infinitely many time differentiable. Given $p = (p_1, \dots, p_n)$ with non-negative integer entries, denote $|p| = p_1 + p_2 + \dots + p_n$ and

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1}\, \partial x_2^{p_2} \cdots \partial x_n^{p_n}}.$$

Let $\{K_n\}_{n=1}^\infty$ be a coutable family of compact sets of Ω satisfying $K_n \subset K_{n+1}$ and $\cup_{n=1}^\infty K_n = \Omega$ (you don't have to show that such a family exists, but would you know how to do it?). Show that the maps $p_{k,n} : C^\infty(\Omega) \to \mathbb{R}$ given by

$$p_{k,n} = \sup_{\substack{|p|=k \\ x\in K_n}} |D^p f(x)|,$$

are well defined and are semi-norms on $C^\infty(\Omega)$. By (d), we obtain a metric. Congratulations, you just showed how to construct a metric on the space of smooth functions! But do you understand its interpretation? What does it mean to say, for example, that the distance of two function in $C^\infty(\Omega)$ is small? Why do we need the sets K_n?

[2]You do not need to prove that such a topology exists, although it does.

Question 7. Finish the proof of the first separation theorem given in class by completing the claims that were indicated as exercises. Notice that two of those claims are questoins 2 and 3 above.

Question 8. Prove the Hölder and Minkowski inequalities as indicated in class, i.e., show that they can be obtained as direct corollaries of the fundamental convex inequality proved in class.

Question 9. Without relying on question 8 or the convex inequality proved in class, establish the following "baby version" of the Hölder and Minkowski inequalities

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}},$$

and

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}},$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$.

Question 10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **convex function**, i.e.,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

$0 \leq t \leq 1$. Let $B_r(0)$ be the open ball of radius $r > 0$ in $\mathbb{R}^n$, and u a continuous function on the closed ball $\overline{B_r(0)}$. Prove Jensen's inequality:

$$f\left(\frac{1}{|B_r(0)|}\int_{B_r(0)} u(x)\, dx\right) \leq \frac{1}{|B_r(0)|}\int_{B_r(0)} f \circ u(x)\, dx,$$

where $|B_r(0)|$ is the volume of $B_r(0)$.