VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

Test 2

NAME: Solutions.

Directions. This exam contains four questions. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	**Points**
1 (25 pts)	
2 (25 pts)	
3 (25 pts)	
4 (25 pts)	
TOTAL	

Question 1. Answer the questions below. Justify your answers.

(a) Is the method of separation of variables guaranteed to always give a solution to a PDE?

(b) What is the difference between a formal solution and an actual solution to a PDE?

(c) Can a formal solution also be a classical solution? Can it be a generalized solution?

(d) Let f be a function defined on $(-L, L)$, $L > 0$, and $F.S.\{f\}$ its Fourier series. Is it true that for any $x \in (-L, L)$ we have that $f(x) = F.S.\{f\}(x)$?

(e) Let $f : (-4, 4) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & -4 < x \leq -2, \\ -1, & -2 < x < 0, \\ x+1, & 0 \leq x \leq 2 \\ x, & 2 < x < 4. \end{cases}$$

Let $F.S.\{f\}$ be its Fourier series. Find $F.S.\{f\}(-2)$, $F.S.\{f\}(-1)$, $F.S.\{f\}(0)$, and $F.S.\{f\}(3)$, i.e., the values of the Fourier series of f at the points $x = -2, -1, 0, 3$.

Solution 1. (a) The method of separation of variables consists in attempting to solve a PDE by supposing that the unknown function is a product of functions of single variables, each of which depends on one of the independent variables of the problem. Thus, if the unknown is $u = u(x_1, \ldots, x_n)$, one tries a solution of the form $u(x_1, \ldots, x_n) = X_1(x_1) \cdots X_n(x_n)$. Since it is based on the educated guess $u(x_1, \ldots, x_n) = X_1(x_1) \cdots X_n(x_n)$ and not every solution to a PDE is of this form, the method is not guaranteed to produce a solution.

(b) A formal solution is an expression that provides a candidate for a solution. It typically consists of a formula involving a series, with no further information on the convergence of the series or other information that makes the expression mathematically well-defined. An actual solution is an expression that solves the PDE pointwise or in the generalized sense.

(c) Yes in both cases. If a formal solution given as a series converges to a C^k function (respectively piece-wise C^k), with k greater or equal to the order of the equation, then the formal solution yields a classical (respectively generalized) solution. The type of convergence involved can vary from problem to problem.

For parts (d) and (e), we recall the following theorem.

Theorem 1. *Let f be a piecewise C^1 function defined on $[-L, L]$. Then, for any $x \in (-L, L)$,*

$$F.S.\{f\}(x) = \frac{1}{2}(f(x^+) + f(x^-)).$$

For $x = \pm L$, the series converges to $\frac{1}{2}(f(-L^+) + f(L^-))$.

(d) No. Take the function

$$f(x) = \begin{cases} -1, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

By the above theorem, $F.S.\{f\}(0) = 0$, but $f(0) = -1$.

(e) Using the above theorem, $F.S.\{f\}(-2) = 0$, $F.S.\{f\}(-1) = -1$, $F.S.\{f\}(0) = 0$, and $F.S.\{f\}(3) = 3$.

Question 2. Consider the following initial-boundary value problem for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } (0,L)\times(0,\infty), \tag{1a}$$

$$u(x,0) = f(x) \quad 0 \le x \le L, \tag{1b}$$

$$u_t(x,0) = g(x) \quad 0 \le x \le L, \tag{1c}$$

$$u(0,t) = 0 \quad t \ge 0, \tag{1d}$$

$$u(L,t) = 0 \quad t \ge 0. \tag{1e}$$

(a) What compatibility conditions do f and g have to satisfy?

(b) Using separation of variables, write two ordinary differential equations that are consequence of equation (1a).

(c) Find a formal solution to the initial-boundary value problem (1).

(d) State sufficient conditions on f and g that guarantee that the formal solution you found in (c) is an actual solution to the problem.

Solution 2. (a) In order to have u well-defined, we need that $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$.

(b) Set $u(x,t) = X(x)T(t)$ and plug into (1a) to find

$$\frac{X''}{X} = \frac{T''}{c^2 T}.$$

Since the left-hand side depends only on x and the right-hand side only on t, both sides need to be equal to a constant λ. Thus

$$X'' = \lambda X, \text{ and } T'' = \lambda c^2 T.$$

(c) This was done in class (see class notes from Oct 3). We find

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos\frac{cn\pi t}{L} + b_n \sin\frac{cn\pi t}{L}) \sin\frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \text{ and } b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

(d) We begin recalling how to show that the formal solution found in (c) is an actual solution. We make an odd extension of f and g to $2L$-periodic functions $\widetilde{f}$ and $\widetilde{g}$ on $\mathbb{R}$. It follows that $\widetilde{f}(-L) = \widetilde{g}(-L) = \widetilde{f}(L) = \widetilde{g}(L) = 0$. Using D'Alembert's formula, we write a solution $\widetilde{u}$ for the wave equation on the real line with initial conditions $\widetilde{f}$ and $\widetilde{g}$. Next, we consider the Fourier series of $\widetilde{u}$, which amounts to consider the Fourier series of $\widetilde{f}$ and $\widetilde{g}$. Because $\widetilde{f}$ is odd, the coefficients a_n and b_n of $F.S.\{\widetilde{u}\}$ agree with the expressions for a_n and b_n in part (c). With trigonometric identities for the sine and cosine of the sum of angles, we expand D'Alemberts formula for $\widetilde{u}$, and observe that the resulting expression agrees with the formal solution u found in (c), and also satisfies the boundary conditions. Therefore, the formal solution in (c) will be an actual solution provided that we can apply theorems for convergence of Fourier series and its derivatives. A theorem for convergence of the Fourier series was stated above, and a theorem for differentiation of Fourier series is the following.

Theorem 2. *Let f be continuous on $[-L, L]$. Suppose that $f(-L) = f(L)$, and that f is piecewise C^2. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. I.e., if*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}),$$

then

$$f'(x) = \sum_{n=1}^{\infty}(a_n(\cos\frac{n\pi x}{L})' + b_n(\sin\frac{n\pi x}{L})'),$$

whenever $f'(x)$ equals its Fourier series. Equivalently,

$$f'(x) = \sum_{n=1}^{\infty}(-a_n\frac{n\pi}{L}\sin\frac{n\pi x}{L} + b_n\frac{n\pi}{L}\cos\frac{n\pi x}{L}).$$

One simple condition guaranteeing the convergence of $F.S.\{\widetilde{u}\}$ and its derivatives on $(0, L)$ is that f and g be smooth.

Question 3. The following questions are about the Fourier transform $\hat{f}$ of a function f.

(a) State conditions that guarantee $\hat{f}$ to be well-defined.

(b) Show that

$$\left(\frac{\partial f}{\partial x_j}\right)^{\hat{}} = ik_j \hat{f}.$$

(c) Show that

$$\mathcal{F}(f * g) = (2\pi)^{\frac{n}{2}} \hat{f}\hat{g},$$

where $f * g$ is the convolution of two functions given by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

Solution 3. (a) A sufficient condition is $\int_{\mathbb{R}^n} |f| < \infty$. Indeed, using that $|e^{-ik\cdot x}| = 1$, we have

$$\left|\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\,dx\right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} |f(x)|\,dx < \infty.$$

(b) We use integration by parts, assuming that $f(x) \to 0$ sufficiently fast when $|x| \to \infty$ so that there are no boundary integrals, to find

$$\begin{aligned}
\left(\frac{\partial f}{\partial x_j}\right)^{\hat{}} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot x}\frac{\partial f}{\partial x_j}\,dx \\
&= -\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} \frac{\partial e^{-ik\cdot x}}{\partial x_j} f(x)\,dx \\
&= ik_j \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\,dx \\
&= ik_j \hat{f}(k).
\end{aligned}$$

(c) Write

$$\begin{aligned}
\mathcal{F}(f * g) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot x}(f * g)(x)\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot x}\int_{\mathbb{R}^n} f(y)g(x-y)\,dy\,dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ik\cdot x} f(y)g(x-y)\,dx\,dy.
\end{aligned}$$

Make the change of variables $z = x - y$ to find

$$\begin{aligned}
\mathcal{F}(f * g) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot y} f(y)\int_{\mathbb{R}^n} e^{-ik\cdot z} g(z)\,dz\,dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ik\cdot y} f(y)\,dy\int_{\mathbb{R}^n} e^{-ik\cdot z} g(z)\,dz \\
&= (2\pi)^{\frac{n}{2}} \hat{f}\hat{g}.
\end{aligned}$$

Question 4. Consider the following initial-value problem:

$$u_{tt} + 2du_t - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{2a}$$

$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{2b}$$

$$u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{2c}$$

where $d > 0$ is a constant.

(a) Applying the Fourier transform in the spatial variable only, show that $\hat{u}$ solves the following problem:

$$\hat{u}_{tt} + 2d\hat{u}_t + |y|^2\hat{u} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{3a}$$

$$\hat{u} = \hat{g} \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{3b}$$

$$\hat{u}_t = \hat{h} \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{3c}$$

(b) Problem (3) is an ODE for $\hat{u}$ for each fixed y. Its solution is (you do not have to show this)

$$\hat{u}(y,t) = \begin{cases} e^{-dt}(\beta_1(y)e^{\gamma(y)t} + \beta_2(y)e^{-\gamma(y)t}) & \text{if } |y| \le d, \\ e^{-dt}(\beta_1(y)e^{i\delta(y)t} + \beta_2(y)e^{-i\delta(y)t}) & \text{if } |y| \ge d, \end{cases} \tag{4}$$

where $\gamma(y) = \sqrt{d^2 - |y|^2}$ with $|y| \le d$, $\delta(y) = \sqrt{|y|^2 - d^2}$ with $|y| \ge d$, and β_1 and β_2 are selected such that

$$\hat{g}(y) = \beta_1(y) + \beta_2(y),$$

and

$$\hat{h}(y) = \begin{cases} \beta_1(y)(\gamma(y) - d) + \beta_2(y)(-\gamma(y) - d) & \text{if } |y| \le d, \\ \beta_1(y)(i\delta(y) - d) + \beta_2(y)(-i\delta - d) & \text{if } |y| \ge d. \end{cases}$$

Using (4), show that the solution to (2) is given by

$$\begin{aligned} u(x,t) &= \frac{e^{-dt}}{\sqrt{2\pi}} \int_{|y| \le d} (\beta_1(y)e^{ixy + \gamma(y)t} + \beta_2(y)e^{ixy - \gamma(y)t})\, dy \\ &+ \frac{e^{-dt}}{\sqrt{2\pi}} \int_{|y| \ge d} (\beta_1(y)e^{i(xy + \delta(y)t)} + \beta_2(y)e^{i(xy - \delta(y)t)})\, dy. \end{aligned}$$

Solution 4. This was a homework problem. See the solutions to HW 5.