VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

Test 1

NAME: Solutions.

Directions. This exam contains four questions. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	**Points**
1 (25 pts)	
2 (25 pts)	
3 (25 pts)	
4 (25 pts)	
TOTAL	

Question 1. (25 pts) For each PDE below, identify the unknown function and state the independent variables. State the order of the PDE. State if the PDE is homogeneous or non-homogeneous, linear or non-linear. (You are asked to state whether the PDE is homogeneous/non-homogeneous and linear/non-linear, not to prove your statement. In particular, you **need not** to write the PDE in the form $F(x, u, \ldots, D^m u) = 0$ or identify the function F.)

(a) $u_{tt} - u_{xx} = f$.

(b) $\cos(t)u_t + x^2 u_x + y^2 u_y = e^{-x^2-y^2}\sin(t)$.

(c) $u_{xx} + u_{yy} = e^u$.

(d) $\displaystyle\sum_{i,j=1}^{n} a^{ij}(x_1, \ldots, x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$.

(e) $\Delta u = u^4$.

Solution 1. (a) Unknown: u. Independent variables: x, t. Order: second. Non-homogeneous PDE. Linear PDE.

(b) Unknown: u. Independent variables: x, y, t. Order: first. Non-homogeneous PDE. Linear PDE.

(c) Unknown: u. Independent variables: x, y. Order: second. Homogeneous PDE. Non-linear PDE.

(d) Unknown: u. Independent variables: x_i, $i = 1, \ldots, n$. Order: second. Homogeneous PDE. Linear PDE.

(e) Unknown: u. Independent variables: x_i, $i = 1, \ldots, n$. Order: second. Homogeneous PDE. Non-linear PDE.

Question 2. (25 pts) Consider the initial-value problem:

$$xu_x + yu_y = 4u, \ -\infty < x < \infty, \ -\infty < y < \infty,$$
$$u = 1 \text{ on the circle } x^2 + y^2 = 1.$$

(a) Identify the PDE and the initial condition.

(b) Find a solution $u = u(x, y)$ for the initial-value problem.

(c) Sketch the projection of the characteristic curves on the xy-plane (i.e., sketch the projected characteristics).

(d) Is the solution you found in (b) unique?

Solution 2. (a) PDE: $xu_x + yu_y = 4u$. Initial condition: $u = 1$ on the circle $x^2 + y^2 = 1$.

(b) Parametrize the initial condition as

$$\Gamma(s) = (x_0(s), y_0(s), u_0(s)) = (\cos s, \sin s, 1), \ 0 \leq s < 2\pi.$$

The system of characteristic equations is

$$\dot{x} = x,$$
$$\dot{y} = y,$$
$$\dot{u} = 4u.$$

The solution is

$$x(t, s) = e^t \cos s, y(t, x) = e^t \sin s, \ u(t, s) = e^{4t}.$$

Compute

$$x^2 + y^2 = (e^t \cos s)^2 + (e^t \sin s)^2 = e^{2t},$$

hence

$$u(x, y) = (x^2 + y^2)^2.$$

(c) We have $y/x = \tan s$, hence the characteristics are straight lines through the origin, see Figure 1.

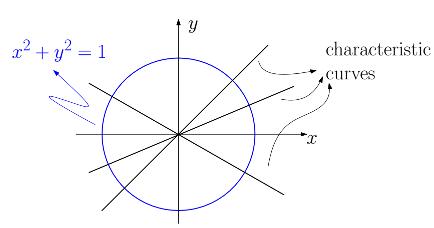

FIGURE 1. Projected characteristics of problem 2c.

(d) Compute

$$J(0, s) = \det \begin{bmatrix} \partial_t x(0, s) & \partial_s x(0, s) \\ \partial_t y(0, s) & \partial_s y(0, s) \end{bmatrix} = \det \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} = 1.$$

Thus, the transversality condition holds and the solution is unique in a neighborhood of Γ.

Question 3. (25 pts) Consider the following initial-value problem:

$$u_t + uu_x = 0 \text{ in } (-\infty, \infty) \times (0, \infty)$$
$$u(x, 0) = h(x),\ -\infty < x < \infty,$$

where h is a given function.

(a) Verify that

$$u(x, t) = h(x - tu(x, t))$$

gives an implicit solution for the initial-value problem.

(b) Show that there exist initial conditions h for which the solution u blows-up (i.e., forms a shock) at a certain finite time t_*, and find a formula for t_*.

(c) Do there exist initial conditions h for which no shock occurs?

Solution 3. (a) Clearly $u(x, 0) = h(x)$. Differentiating $u(x, t) = h(x - tu(x, t))$ with respect to t gives

$$u_t = h'(x - tu)(-u - tu_t),$$

and differentiating with respect to x produces.

$$u_x = h'(x - tu)(1 - tu_x).$$

Hence

$$u_t + uu_x = -th'(x - tu)(u_t + uu_x),$$

or

$$(1 + th'(x - tu))(u_t + uu_x) = 0.$$

Hence $u_t + uu_x = 0$ as long as $1 + th'(x - tu) \neq 0$. Since $1 + th'(x - tu) = 0$ is the condition for shocks, we have shown that the given formula defines an implicit solution as long as shocks do not occur.

(b) From the formula $u_x = h'(x - tu)(1 - tu_x)$ computed above we obtain

$$u_x = \frac{h'}{1 + th'}.$$

Hence, for h such that $h'(s) < 0$ for some $s \in \mathbb{R}$, we obtain that u_x blows-up at time

$$t_* = -\frac{1}{h'(s)}.$$

(c) If $h'(s) > 0$ for all s, then no blow-up occurs (recall that $t \geq 0$). From the formula $(1 + th'(x - tu))(u_t + uu_x) = 0$ derived in (a), we also see that u remains a solution for all $t \geq 0$.

Question 4. (25 pts) Consider the following initial-value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \text{ in } (-\infty, \infty) \times (0, \infty), \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned} \tag{1}$$

(a) Solve (1) when $f(x) = x^2$ and $g(x) = 0$.

(b) Assume now that $c = 1$ and

$$f(x) = \begin{cases} 1, & -2 \le x \le 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} -1, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Draw a diagram in the (x,t)-plane indicating the different regions where the solution is influenced by the initial condition f and g and the regions where the solution is identically zero. (This is similar to what was done in class. You do **not** have to find u.)

Solution 4. (a) By D'Alembert's formula:

$$u(x,t) = \frac{(x+ct)^2 + (x-ct)^2}{2}.$$

(b) The regions are summarized in Figure 2.

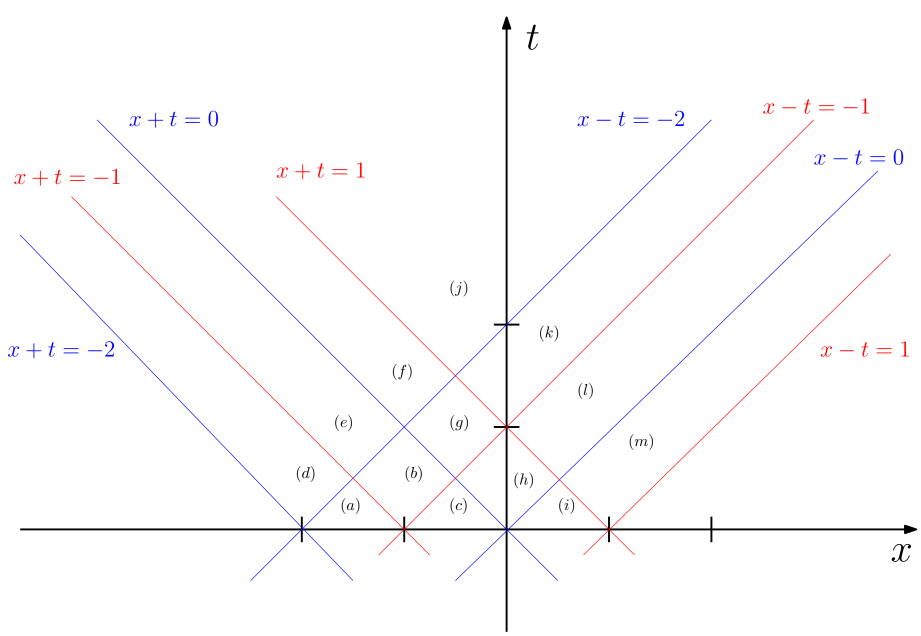

FIGURE 2. Problem 4b.