VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

Practice problems for Test 2

Test 2 will cover material from Oct 3 to Oct 19. See the course webpage for the material covered over this period. The Laplace transform will not be in the test.

Question 1. Answer the questions below. Justify your answers.

(a) What is the method of separation of variables? Is it guaranteed to always produce a solution to a PDE?

(b) What is the difference between a formal solution and an actual solution to a PDE?

(c) Can a formal solution also be a classical solution? Can it be a generalized solution?

(d) Let f be a function defined on $(-L, L)$, $L > 0$, and $F.S.\{f\}$ its Fourier series. Is it true that for any $x \in (-L, L)$ we have that $f(x) = F.S.\{f\}(x)$?

(e) Let $f : (-2, 2) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & -2 < x \leq -1, \\ 2, & -1 < x < 0, \\ 1, & 0 \leq x \leq 1 \\ x, & 1 < x < 2. \end{cases}$$

Let $F.S.\{f\}$ be its Fourier series. Find $F.S.\{f\}(-1)$, $F.S.\{f\}(0)$, $F.S.\{f\}(1)$, and $F.S.\{f\}(1.5)$, i.e., the values of the Fourier series of f at the points $x = -1, 0, 1, 1.5$. *Hint:* you do not need to compute the Fourier series of f to solve this problem.

Solution 1. (a) The method of separation of variables consists in attempting to solve a PDE by supposing that the unknown function is a product of functions of single variables, each of which depends on one of the independent variables of the problem. Thus, if the unknown is $u = u(x_1, \dots, x_n)$, one tries a solution of the form $u(x_1, \dots, x_n) = X_1(x_1) \cdots X_n(x_n)$. The method is not guaranteed to produce a solution.

(b) A formal solution is an expression that provides a candidate for a solution. It typically consists of a formula involving a series, with no further information on the convergence of the series or other information that makes the expression mathematically well-defined.

(c) Yes in both cases. If a formal solution given as a series converges to a C^k function (respectively piece-wise C^k), with k greater or equal to the order of the equation, then the formal solution yields a classical (respectively generalized) solution. The type of convergence involved can vary from problem to problem.

For parts (d) and (e), we recall the following theorem.

Theorem 1. *Let f be a piecewise C^1 function defined on $[-L, L]$. Then, for any $x \in (-L, L)$,*

$$F.S.\{f\}(x) = \frac{1}{2}(f(x^+) + f(x^-)).$$

For $x = \pm L$, *the series converges to* $\frac{1}{2}(f(-L^+) + f(L^-))$.

(d) No. Take the function

$$f(x) = \begin{cases} -1, & x \le 0, \\ 1, & x > 0. \end{cases}$$

By the above theorem, $F.S.\{f\}(0) = 0$, but $f(0) = -1$.

(e) Using the above theorem, $F.S.\{f\}(-1) = (-1+2)/2 = 1/2$, $F.S.\{f\}(0) = (2+1)/2 = 3/2$, $F.S.\{f\}(1) = (1+1)/2 = 1$, and $F.S.\{f\}(1.5) = 1.5$.

Question 2. Consider the following initial-boundary value problem for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0 \qquad \text{in } (0, L) \times (0, \infty), \tag{1a}$$

$$u(x, 0) = f(x) \quad 0 \le x \le L, \tag{1b}$$

$$u_t(x, 0) = g(x) \quad 0 \le x \le L, \tag{1c}$$

$$u(0, t) = 0 \qquad t \ge 0, \tag{1d}$$

$$u(L, t) = 0 \qquad t \ge 0. \tag{1e}$$

(a) What compatibility conditions do f and g have to satisfy?

(b) Using separation of variables, write two ordinary differential equations that are consequence of equation (1a).

(c) Find a formal solution to the initial-boundary value problem (1).

(d) State sufficient conditions on f and g that guarantee that the formal solution you found in (c) is an actual solution to the problem.

(e) Explain how a formal solution to (1) can be showed to be an actual solution under the conditions you stated in (d). You are not required to provide a formal proof. Rather, outline the argument and its main steps. In doing so, state any relevant theorems you need to invoke.

Solution 2. (a) In order to have u well-defined, we need that $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$.

(b) Set $u(x, t) = X(x)T(t)$ and plug into (1a) to find

$$\frac{X''}{X} = \frac{T''}{c^2 T}.$$

Since the left-hand side depends only on x and the right-hand side only on t, both sides need to be equal to a constant λ. Thus

$$X'' = \lambda X, \text{ and } T'' = \lambda c^2 T.$$

(c) This was done in class. We find

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L}) \sin \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \text{ and } b_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx.$$

(d) and (e) We answer parts (d) and (e) together. We make an odd extension of f and g to $2L$-periodic functions $\widetilde{f}$ and $\widetilde{g}$ on $\mathbb{R}$. It follows that $\widetilde{f}(-L) = \widetilde{g}(-L) = \widetilde{f}(L) = \widetilde{g}(L) = 0$. Using

D'Alembert's formula, we write a solution $\widetilde{u}$ for the wave equation on the real line with initial conditions $\widetilde{f}$ and $\widetilde{g}$. Next, we consider the Fourier series of $\widetilde{u}$, which amounts to consider the Fourier series of $\widetilde{f}$ and $\widetilde{g}$. Because $\widetilde{f}$ is odd, the coefficients a_n and b_n of $F.S.\{\widetilde{u}\}$ agree with the expressions for a_n and b_n in part (c). With trigonometric identities for the sine and cosine of the sum of angles, we expand D'Alemberts formula for $\widetilde{u}$, and observe that the resulting expression agrees with the formal solution u found in (c), and also satisfies the boundary conditions. Therefore, the formal solution in (c) will be an actual solution provided that we can apply theorems for convergence of Fourier series and its derivatives. A theorem for convergence of the Fourier series was stated above, and a theorem for differentiation of Fourier series is the following.

Theorem 2. *Let f be continuous on $[-L, L]$. Suppose that $f(-L) = f(L)$, and that f is piecewise C^2. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. I.e., if*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}),$$

then

$$f'(x) = \sum_{n=1}^{\infty}(a_n(\cos\frac{n\pi x}{L})' + b_n(\sin\frac{n\pi x}{L})'),$$

whenever $f'(x)$ equals its Fourier series. Equivalently,

$$f'(x) = \sum_{n=1}^{\infty}(-a_n\frac{n\pi}{L}\sin\frac{n\pi x}{L} + b_n\frac{n\pi}{L}\cos\frac{n\pi x}{L}).$$

One simple condition guaranteeing the convergence of $F.S.\{\widetilde{u}\}$ and its derivatives on $(0, L)$ is that f and g be smooth.

See the class notes for a detailed presentation of the above argument.

Question 3. Consider the following initial-boundary value problem for the heat equation:

$$\begin{aligned} u_t - ku_{xx} &= 0 && \text{in } (0, L) \times (0, \infty), && (2a)\\ u(x, 0) &= f(x) && 0 \le x \le L, && (2b)\\ u(0, t) &= 0 && t \ge 0, && (2c)\\ u(L, t) &= 0 && t \ge 0. && (2d) \end{aligned}$$

(a) The following expression is a formal solution to problem (2) (you do not need to establish this):

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}, \tag{3}$$

where

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx.$$

What happens to the formal solution when $t \to \infty$? How do you interpret this result?

(b) Determine the formal solution (3) when $k = 1$, $L = \pi$, and

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{2}, \\ 2, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

(c) Prove that for any fixed $t > 0$, the formal solution you found in (b) converges for any $x \in [0, \pi]$.

Solution 3. (a) When $t \to \infty$, the formal expression gives that $u(x,t) \to 0$, meaning that the temperature of the rod goes to zero (recall the physical interpretation of the heat equations discussed at the beginning of the course). This makes sense in light of the boundary conditions: we are considering the case where the rod's endpoints are kept at zero temperature and no other heat exchange with the environment is allowed. Hence, the rod's temperature will eventually becomes zero.

(b) We find

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = \frac{4}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin(nx)\,dx = \frac{4}{n\pi}((-1)^{n+1} + \cos\frac{n\pi}{2}),$$

and the solution is (3) with this expression for b_n, $L = \pi$, and $k = 1$.

(c) Since the exponential increases faster than any polynomial, we have that, if $t > 0$ is fixed, then

$$\left| \frac{4}{n\pi}((-1)^{n+1} + \cos\frac{n\pi}{2})e^{-n^2 t} \right| \leq \frac{C}{n^2},$$

for some constant $C > 0$ and for all n sufficiently large. Since the series of $\frac{1}{n^2}$ converges, so does the formal solution.

The questions below deal with the Fourier transform. In doing so, you can use the following property:

$$(\hat{f})\check{} = f, \tag{4}$$

which you need not to prove. The properties below (as well as (4)) have been stated in class. You are referred to the class notes for further details on the notation.

Prove the following properties of the Fourier transform.

Question 4. The Fourier transform and its inverse are linear.

Solution 4. This follows from the linearity of the integral.

Question 5.

$$\left(\frac{\partial f}{\partial x_j}\right)\hat{} = ik_j \hat{f},$$

and

$$\left(\frac{\partial^{m_1+\cdots+m_n} f}{\partial x_1^{m_1}\cdots \partial x_n^{m_n}}\right)\hat{} = (ik_1)^{m_1}\cdots(ik_n)^{m_n}\hat{f}.$$

Solution 5. The first formula follows from integration by parts. The second formula follows from applying the first one repeated times.

Question 6. $\overline{\hat{f}(k)} = \hat{f}(-k)$ if f is a real function.

Solution 6. This follows directly from taking the complex conjugate of the Fourier transform.

Question 7. $\mathcal{F}(f * g) = (2\pi)^{\frac{n}{2}}\hat{f}\hat{g}$.

Solution 7. This follows from a direct change of variables followed by a change of order of integration.

For questions 8 to 10, assume that $n = 1$.

Question 8. $\mathcal{F}(f(x-a)) = e^{-iak}\hat{f}(k)$.

Question 9. $\mathcal{F}(e^{iax}f(x)) = \hat{f}(k-a)$.

Question 10. $\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}(\frac{k}{a})$, $a \neq 0$.

Question 11. Show that the Fourier transform of $e^{-a|x|}$, $a > 0$, $x \in \mathbb{R}$, is $\frac{2a}{a^2+k^2}$.

Solution 8-10. The solution to questions (8)-(10) follows by direct integration.

Question 12. Review your class notes and the posted solutions to the HW problems.