VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

Practice problems for Test 2

Test 2 will cover material from Oct 3 to Oct 19. See the course webpage for the material covered over this period. The Laplace transform will not be in the test.

Question 1. Answer the questions below. Justify your answers.

(a) What is the method of separation of variables? Is it guaranteed to always produce a solution to a PDE?

(b) What is the difference between a formal solution and an actual solution to a PDE?

(c) Can a formal solution also be a classical solution? Can it be a generalized solution?

(d) Let f be a function defined on $(-L, L)$, $L > 0$, and $F.S.\{f\}$ its Fourier series. Is it true that for any $x \in (-L, L)$ we have that $f(x) = F.S.\{f\}(x)$?

(e) Let $f : (-2, 2) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & -2 < x \leq -1, \\ 2, & -1 < x < 0, \\ 1, & 0 \leq x \leq 1 \\ x, & 1 < x < 2. \end{cases}$$

Let $F.S.\{f\}$ be its Fourier series. Find $F.S.\{f\}(-1)$, $F.S.\{f\}(0)$, $F.S.\{f\}(1)$, and $F.S.\{f\}(1.5)$, i.e., the values of the Fourier series of f at the points $x = -1, 0, 1, 1.5$. *Hint:* you do not need to compute the Fourier series of f to solve this problem.

Question 2. Consider the following initial-boundary value problem for the wave equation:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 && \text{in } (0, L) \times (0, \infty), && \text{(1a)} \\ u(x, 0) &= f(x) && 0 \leq x \leq L, && \text{(1b)} \\ u_t(x, 0) &= g(x) && 0 \leq x \leq L, && \text{(1c)} \\ u(0, t) &= 0 && t \geq 0, && \text{(1d)} \\ u(L, t) &= 0 && t \geq 0. && \text{(1e)} \end{aligned}$$

(a) What compatibility conditions do f and g have to satisfy?

(b) Using separation of variables, write two ordinary differential equations that are consequence of equation (1a).

(c) Find a formal solution to the initial-boundary value problem (1).

(d) State sufficient conditions on f and g that guarantee that the formal solution you found in (c) is an actual solution to the problem.

(e) Explain how a formal solution to (1) can be showed to be an actual solution under the conditions you stated in (d). You are not required to provide a formal proof. Rather, outline the argument and its main steps. In doing so, state any relevant theorems you need to invoke.

Question 3. Consider the following initial-boundary value problem for the heat equation:

$$
\begin{aligned}
u_t - k u_{xx} &= 0 && \text{in } (0,L)\times(0,\infty), && (2a)\\
u(x,0) &= f(x) && 0 \le x \le L, && (2b)\\
u(0,t) &= 0 && t \ge 0, && (2c)\\
u(L,t) &= 0 && t \ge 0. && (2d)
\end{aligned}
$$

(a) The following expression is a formal solution to problem (2) (you do not need to establish this):

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}, \qquad (3)$$

where

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

What happens to the formal solution when $t \to \infty$? How do you interpret this result?

(b) Determine the formal solution (3) when $k = 1$, $L = \pi$, and

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{2}, \\ 2, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

(c) Prove that for any fixed $t > 0$, the formal solution you found in (b) converges for any $x \in [0,\pi]$.

The questions below deal with the Fourier transform. In doing so, you can use the following property:

$$(\hat{f})\check{} = f, \qquad (4)$$

which you need not to prove. The properties below (as well as (4)) have been stated in class. You are referred to the class notes for further details on the notation.

Prove the following properties of the Fourier transform.

Question 4. The Fourier transform and its inverse are linear.

Question 5.

$$\left(\frac{\partial f}{\partial x_j}\right)\hat{} = i k_j \hat{f},$$

and

$$\left(\frac{\partial^{m_1+\cdots+m_n} f}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\right)\hat{} = (ik_1)^{m_1}\cdots(ik_n)^{m_n}\hat{f}.$$

Question 6. $\overline{\hat{f}(k)} = \hat{f}(-k)$ if f is a real function.

Question 7. $\mathcal{F}(f * g) = (2\pi)^{\frac{n}{2}}\hat{f}\hat{g}$.

For questions 8 to 10, assume that $n = 1$.

Question 8. $\mathcal{F}(f(x-a)) = e^{-iak}\hat{f}(k)$.

Question 9. $\mathcal{F}(e^{iax}f(x)) = \hat{f}(k-a)$.

Question 10. $\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}(\frac{k}{a})$, $a \neq 0$.

Question 11. Show that the Fourier transform of $e^{-a|x|}$, $a > 0$, $x \in \mathbb{R}$, is $\frac{2a}{a^2+k^2}$.

Question 12. Review your class notes and the posted solutions to the HW problems.