VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 3

Question 1. Consider the Cauchy problem for Burger's equation:

$$u_t + uu_x = 0,$$
$$u(x, 0) = h(x),$$

for $(x, t) \in (-\infty, \infty) \times (0, \infty)$.

(a) Find conditions on h that guarantee that no shock waves will form.

(b) Derive a necessary condition for the formation of a shock wave.

Question 2. Consider the eikonal equation:

$$u_x^2 + u_y^2 = n^2, \tag{1}$$

where $n = n(x, y)$ is a given function. The eikonal equation has important applications in optics.

The goal of this problem is to show how the method of characteristics can be used to solve the eikonal equation, which is a fully non-linear first order PDE.

Assume that an initial condition for (1) is given in the form of a parametrized curve $\Gamma(s) = (x_0(s), y_0(s), u_0(s))$.

(a) Show that (1) is equivalent to $(u_x, u_y, n^2) \cdot (u_x, u_y, -1) = 0$ and interpret this geometrically.

(b) Using (a), explain why it makes sense to consider the following system of characteristic equations for $x = x(t, s)$, $y = y(t, s)$, and $u = u(t, s)$ (recall the geometric meaning of the characteristic curves)

$$\dot{x} = u_x \tag{2a}$$
$$\dot{y} = u_y \tag{2b}$$
$$\dot{u} = n^2 \tag{2c}$$

(c) From equations (2) and (1), derive

$$\ddot{x} = \frac{1}{2}\partial_x n^2 \tag{3a}$$
$$\ddot{y} = \frac{1}{2}\partial_y n^2 \tag{3b}$$
$$\dot{u} = n^2 \tag{3c}$$

(d) Show that the solution to (1) is given by

$$u(x(t, s), y(t, s)) = u(x_0(s), y_0(s)) + \int_0^t (n(x(\tau, s), y(\tau, s)))^2 \, d\tau,$$

where $(x(\tau, s), y(\tau, s))$ is a solution to (3a)-(3b).

Question 3. Solve (1) when $n(x, y) = 1$ and with initial condition $u = 1$ on the curve $y = 2x$.

Question 4. Consider

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \text{ in } (-\infty,\infty)\times(0,\infty), \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned} \tag{4}$$

where $c = 3$ and

$$f(x) = g(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2. \end{cases}$$

(a) Without finding a general formula for u, find $u(0,2)$.

(b) Without finding a general formula for u,compute

$$\lim_{t\to\infty} u(x,t).$$

(c) Solve (4).

(d) Is the solution you found classical? Explain.

Question 5. Consider the following problem for the wave equation on the half-line, i.e., for $x \ge 0$ rather than $-\infty < x < \infty$.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 \text{ in } (0,\infty)\times(0,\infty), \\ u(x,0) &= x^2 \text{ for } 0 \le x < \infty, \\ u_t(x,0) &= 6x \text{ for } 0 \le x < \infty, \\ u(0,t) &= t^2 \text{ for } t > 0. \end{aligned} \tag{5}$$

(a) Notice that now we have the condition $u(0,t) = t^2$ for $t > 0$, which was absent when $-\infty < x < \infty$. Explain why such a condition was introduced.

(b) Solve (5).

Question 6. This problem shows how one could have "guessed" that solutions to the wave equation are a sum of a forward and a backward wave. Consider

$$u_{tt} - c^2 u_{xx} = 0 \text{ in } (0,\infty)\times(-\infty,\infty). \tag{6}$$

Define the change of variables $\alpha = \alpha(t,x) = x+ct$ and $\beta = \beta(t,x) = x-ct$, and set $v(\alpha,\beta) = u(t,x)$, i.e.,

$$u(t,x) = v(\alpha(t,x), \beta(t,x)).$$

(a) Show that (6) is equivalent to $\partial_\alpha \partial_\beta v = 0$.

(b) Use part (a) to conclude that $u(t,x) = F(x+ct) + G(x-ct)$.