VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 3 Solutions

Question 1. Consider the Cauchy problem for Burger's equation:

$$u_t + uu_x = 0,$$
$$u(x,0) = h(x),$$

for $(x,t) \in (-\infty,\infty) \times (0,\infty)$.

(a) Find conditions on h that guarantee that no shock waves will form.

(b) Derive a necessary condition for the formation of a shock wave.

Solution 1. In class, we derived the relation

$$u_x = \frac{h'}{1+th'},$$

from which we conclude that u_x blows-up when the denominator on the right-hand side vanishes. Since $t \geq 0$, if h is never decreasing, so that $h' \geq 0$, then no blow-up occurs. We also see that a necessary condition for shock formation is that $h'(x) < 0$ for at least one x.

Question 2. Consider the eikonal equation:

$$u_x^2 + u_y^2 = n^2, \tag{1}$$

where $n = n(x,y)$ is a given function. The eikonal equation has important applications in optics.

The goal of this problem is to show how the method of characteristics can be used to solve the eikonal equation, which is a fully non-linear first order PDE.

Assume that an initial condition for (1) is given in the form of a parametrized curve $\Gamma(s) = (x_0(s), y_0(s), u_0(s))$.

(a) Show that (1) is equivalent to $(u_x, u_y, n^2) \cdot (u_x, u_y, -1) = 0$ and interpret this geometrically.

(b) Using (a), explain why it makes sense to consider the following system of characteristic equations for $x = x(t,s)$, $y = y(t,s)$, and $u = u(t,s)$ (recall the geometric meaning of the characteristic curves)

$$\dot{x} = u_x \tag{2a}$$
$$\dot{y} = u_y \tag{2b}$$
$$\dot{u} = n^2 \tag{2c}$$

(c) From equations (2) and (1), derive

$$\ddot{x} = \frac{1}{2}\partial_x n^2 \tag{3a}$$
$$\ddot{y} = \frac{1}{2}\partial_y n^2 \tag{3b}$$
$$\dot{u} = n^2 \tag{3c}$$

(d) Show that the solution to (1) is given by

$$u(x(t,s),y(t,s)) = u(x_0(s),y_0(s)) + \int_0^t (n(x(\tau,s),y(\tau,s)))^2\,d\tau,$$

where $(x(\tau,s),y(\tau,s))$ is a solution to (3a)-(3b).

Solution 2. Computing $(u_x,u_y,n^2)\cdot(u_x,u_y,-1)$ we see that (1) is equivalent to $(u_x,u_y,n^2)\cdot(u_x,u_y,-1)=0$. Since $(u_x,u_y,-1)$ is normal to the graph of u, we see that (u_x,u_y,n^2) must be tangent to it. As the characteristic equations correspond to equations for curves lying on the graph of u, we see that we should consider (2).

Using the chain rule and equation (2a), we find

$$\begin{aligned}\ddot{x} = \frac{d}{dt}\dot{x} &= u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt}\\ &= u_{xx}u_x + u_{xy}u_y = \frac{1}{2}\partial_x(u_x^2+u_y^2)\\ &= \frac{1}{2}\partial_x n^2,\end{aligned}$$

which is (3a). Similarly we obtain (3b).

Finally, from our definitions and the chain rule we have that

$$\begin{aligned}\frac{du}{dt} &= u_x\frac{dx}{dt} + u_y\frac{dy}{dt}\\ &= u_x^2+u_y^2\\ &= n^2.\end{aligned}$$

Integrating in t yields the final answer.

Question 3. Solve (1) when $n(x,y)=1$ and with initial condition $u=1$ on the curve $y=2x$.

Solution 3. Parametrize the initial condition as

$$x_0(s)=s, y_0(s)=2s, u_0(s)=1.$$

Since (3a) and (3b) are second order ODEs, we need initial conditions for $\dot{x}$ and $\dot{y}$ as well, which we denote $\dot{x}_0(s)$ and $\dot{y}_0(s)$. From (2a), (2b), and (1) we know that

$$(\dot{x}_0)^2 + (\dot{y}_0)^2 = n^2 = 1. \tag{4}$$

Differentiating $u_0(s)=1$ with respect to s and using (2a)-(2b) produces

$$\dot{x}_0 + 2\dot{y}_0 = 0. \tag{5}$$

Solving (4)-(5) yields

$$\dot{x}_0 = \frac{2}{\sqrt{5}}, \dot{y}_0 = -\frac{1}{\sqrt{5}}.$$

We can now solve (3a)-(3b) with the above initial conditions to find

$$x(t,s) = \frac{2}{\sqrt{5}}t + s, y(t,s) = -\frac{1}{\sqrt{5}}t + 2s.$$

Using these expressions in the formula for u gives

$$u(t,s) = t+1.$$

We can solve for (t, s) in terms of (x, y) to finally obtain

$$u(x, y) = 1 + \frac{2x - y}{\sqrt{5}}.$$

Question 4. Consider

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \text{ in } (-\infty, \infty) \times (0, \infty), \\ u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x), \end{aligned} \tag{6}$$

where $c = 3$ and

$$f(x) = g(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2. \end{cases}$$

(a) Without finding a general formula for u, find $u(0, 2)$.

(b) Without finding a general formula for u,compute

$$\lim_{t \to \infty} u(x, t).$$

(c) Solve (6).

(d) Is the solution you found classical? Explain.

Solution 4. Recall D'Alembert's formula:

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds. \tag{7}$$

Using (7) with $x = 0$, $t = 2$, and $c = 3$, we find

$$u(0, 2) = \frac{2}{3}.$$

Since

$$\lim_{t \to \infty} f(x + 3t) = 0 = \lim_{t \to \infty} f(x - 3t),$$

and

$$\lim_{t \to \infty} \frac{1}{2 \cdot 3} \int_{x-3t}^{x+3t} g(s)\, ds = \frac{1}{6} \int_{-\infty}^{\infty} g(s)\, ds = \frac{1}{6} \int_{-2}^{2} g(s)\, ds = \frac{2}{3},$$

we find $\lim_{t \to \infty} u(x, t) = \frac{2}{3}$.

Using (7) and arguing as in class we find

$$u(x, t) = \begin{cases} 1 + t, & -2 \le x + 3t \le 2,\ -2 \le x - 3t \le 2,\ t \ge 0, \\ \frac{1}{2} + \frac{x+3t+2}{6}, & -2 \le x + 3t \le 2,\ x - 3t < -2,\ t \ge 0, \\ \frac{1}{2} + \frac{2-(x-3t)}{6}, & 2 < x + 3t,\ -2 \le x - 3t \le 2,\ t \ge 0, \\ \frac{2}{3}, & 2 < x + 3t,\ x - 3t < -2,\ t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

The solution has singularities and is piecewise C^2, hence it is a generalized solution.

Question 5. Consider the following problem for the wave equation on the half-line, i.e., for $x \geq 0$ rather than $-\infty < x < \infty$.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 \ \text{ in } \ (0,\infty)\times(0,\infty), \\ u(x,0) &= x^2 \text{ for } 0 \leq x < \infty, \\ u_t(x,0) &= 6x \text{ for } 0 \leq x < \infty, \\ u(0,t) &= t^2 \text{ for } t > 0. \end{aligned} \tag{8}$$

(a) Notice that now we have the condition $u(0,t) = t^2$ for $t > 0$, which was absent when $-\infty < x < \infty$. Explain why such a condition was introduced.

(b) Solve (8).

Solution 5. The line $x = 0$ corresponds to a boundary, hence a boundary condition needs to be given. That is why we have $u(0,t) = t^2$.

For this problem, it is instructive to consider the general situation

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 0 \ \text{ in } \ (0,\infty)\times(0,\infty), \\ u(x,0) &= f(x) \text{ for } 0 < x < \infty, \\ u_t(x,0) &= g(x) \text{ for } 0 < x < \infty, \\ u(0,t) &= h(t) \text{ for } t > 0, \end{aligned} \tag{9}$$

where f, g, and h are given functions. Since the equation is homogeneous, by linearity, the solution to (9) can be written as

$$u = v + w,$$

where v solves

$$\begin{aligned} v_{tt} - c^2v_{xx} &= 0 \ \text{ in } \ (0,\infty)\times(0,\infty), \\ v(x,0) &= f(x) \text{ for } 0 < x < \infty, \\ v_t(x,0) &= g(x) \text{ for } 0 < x < \infty, \\ v(0,t) &= 0 \text{ for } t > 0, \end{aligned} \tag{10}$$

and w solves

$$\begin{aligned} w_{tt} - c^2w_{xx} &= 0 \ \text{ in } \ (0,\infty)\times(0,\infty), \\ w(x,0) &= 0 \text{ for } 0 < x < \infty, \\ w_t(x,0) &= 0 \text{ for } 0 < x < \infty, \\ w(0,t) &= h(t) \text{ for } t > 0. \end{aligned} \tag{11}$$

We start solving (10). Since we have a formula for the solution in the case $-\infty < x < \infty$, it is natural to extend the problem to the entire real line $\mathbb{R}$, solve it there, and then restrict to $x > 0$ to obtain a solution to (10). The crucial question is how to extend the problem to $\mathbb{R}$. Since $v(0,t) = 0$, we expect $v(0,0) = 0$. Thus, we extend f and g as odd functions, as an odd function necessarily vanishes at the origin. More precisely, define

$$\widetilde{f}(x) = \begin{cases} f(x), & x > 0, \\ 0, & x = 0, \\ -f(-x), & x < 0, \end{cases}$$

and

$$\widetilde{g}(x) = \begin{cases} g(x), & x > 0, \\ 0, & x = 0, \\ -g(-x), & x < 0. \end{cases}$$

We now solve the problem

$$\begin{aligned} \widetilde{v}_{tt} - c^2 \widetilde{v}_{xx} &= 0 \text{ in } (-\infty, \infty) \times (0, \infty), \\ \widetilde{v}(x, 0) &= \widetilde{f}(x) \text{ for } -\infty < x < \infty, \\ \widetilde{v}_t(x, 0) &= \widetilde{g}(x) \text{ for } -\infty < x < \infty, \end{aligned}$$

which can be done by a direct application of D'Alembert's formula. You can verify that since the initial conditions are odd functions, so will be the solution $\widetilde{v}$, thus $\widetilde{v}$ satisfies $\widetilde{v}(0, t) = 0$. We now obtain the solution v to (10) upon setting

$$v(x, t) = \widetilde{v}(x, t) \text{ for } x \geq 0, t \geq 0.$$

Next, we move to solve (11). First, notice that D'Alemberts formula remains valid for $x \geq ct$. Since the initial conditions are zero, we conclude that

$$w(x, t) = 0 \text{ for } x \geq ct.$$

Now assume that $0 \leq x < ct$. We know that w can be written as

$$w(x, t) = F(x + ct) + G(x - ct). \tag{12}$$

Setting $x = 0$ and using the boundary condition,

$$w(0, t) = F(ct) + G(-ct) = h(t),$$

which gives, setting $z = -ct$,

$$F(-z) + G(z) = h(-\frac{z}{c}).$$

Plugging now $z = x - ct$ produces

$$G(x - ct) = h(t - \frac{x}{c}) - F(-x + ct).$$

Using this into (12):

$$w(x, t) = h(t - \frac{x}{c}) + F(x + ct) - F(-x + ct).$$

But recall that w vanishes for $x \geq ct$, so in particular along the line $x = ct$. Thus, by continuity we must have

$$w(x, \frac{x}{c}) = h(0) + F(2x) - F(0) = 0.$$

From the initial conditions we get $h(0) = 0$ and $F(0) = 0$ (recall that $F(x + ct) = (w(x + ct, 0) + w(x - ct, 0))/2$ for $x \geq ct$), thus $F = 0$. We conclude that

$$w(x, t) = h(t - \frac{x}{c}) \text{ for } x < ct.$$

Thus,

$$w(x, t) = \begin{cases} 0, & x \geq ct, \\ h(t - \frac{x}{c}) & x < ct. \end{cases}$$

Remark. Notice that f, g, and h must satisfy some compatibility conditions (which we implicitly used above). Can you identify them?

Question 6. This problem shows how one could have "guessed" that solutions to the wave equation are a sum of a forward and a backward wave. Consider

$$u_{tt} - c^2 u_{xx} = 0 \text{ in } (0,\infty) \times (-\infty,\infty). \tag{13}$$

Define the change of variables $\alpha = \alpha(t,x) = x+ct$ and $\beta = \beta(t,x) = x-ct$, and set $v(\alpha,\beta) = u(t,x)$, i.e.,

$$u(t,x) = v(\alpha(t,x), \beta(t,x)).$$

(a) Show that (13) is equivalent to $\partial_\alpha \partial_\beta v = 0$.

(b) Use part (a) to conclude that $u(t,x) = F(x+ct) + G(x-ct)$.

Solution 6. (a) Using the chain rule we find

$$u_t = v_\alpha \alpha_t + v_\beta \beta_t = cv_\alpha - cv_\beta$$

and

$$u_{tt} = c^2(v_{\alpha\alpha} - 2v_{\alpha\beta} + v_{\beta\beta}).$$

Doing a similar calculation for u_x and u_{xx} and plugging into (13) gives

$$0 = -4c^2 v_{\alpha\beta},$$

thus $v_{\alpha\beta} = 0$.

(b) Since $v_{\alpha\beta} = 0$, we conclude that v_α is independent of β, so $v_\alpha(\alpha,\beta) = f(\alpha)$ for some function f. Integrating with respect to α gives

$$v(\alpha,\beta) = \int f(\alpha)\, d\alpha + G(\beta),$$

for some function G. Writing $F(\alpha) = \int f(\alpha)\, d\alpha$ and using the definition of α, β gives the result.