VANDERBILT UNIVERSITY

MATH 41140 – PARTIAL DIFFERENTIAL EQUATIONS

HW 2

Question 1. Solve the following problems. In each case, sketch the characteristic curves, and indicate the region in the xy-plane where the solution is defined.

(a) $xu_y - yu_x = u$,

with the condition $u(x,0) = g(x)$, where g is a given function.

(b) $u_x + u_y = u^2$,

for (x,y) in the region $\{y \geq 0\}$, with the condition $u(x,0) = g(x)$, where g is a given function. Find the solution in the case $g(x) = x^2$.

(c) $u_x + u_y + u = 1$,

with the condition $u = \sin x$ on $y = x + x^2$, $x > 0$.

Question 2. Derive the system of characteristic equations for the quasilinear equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u).$$

You can follow closely what was done in class for the linear case.

Question 3. Solve

$$uu_x - uu_y = u^2 + (x+y)^2,$$

with initial condition $u(x,0) = 1$. (*Hint:* after writing the characteristic equations, identify an equation satisfied by $x+y$.)

Question 4. As we discussed in class, the method of characteristics requires solving a system of ODEs, the characteristic equations. Therefore, it is important to know when the characteristic equations admit solutions and when such solutions are unique. Review your notes/textbook from ODEs and identify important theorems that guarantee when solutions to systems of ODEs exist and are unique. State at least one such theorem. You can consult, for instance:

- Fundamentals of differential equations and boundary value problems, by Nagle, Saff, and Sinder, chapter 13.
- Ordinary differential equations, by Hartman, chapters II and III.