VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 2 Solutions

Question 1. Solve the following problems. In each case, sketch the characteristic curves, and indicate the region in the xy-plane where the solution is defined.

(a) $xu_y - yu_x = u$,

with the condition $u(x,0) = g(x)$, where g is a given function.

(b) $u_x + u_y = u^2$,

for (x,y) in the region $\{y \geq 0\}$, with the condition $u(x,0) = g(x)$, where g is a given function. Find the solution in the case $g(x) = x^2$.

(c) $u_x + u_y + u = 1$,

with the condition $u = \sin x$ on $y = x + x^2$, $x > 0$.

Solution 1. (a) Parametrize the initial condition by

$$x_0(s) = s, y_0(s) = 0, u_0(s) = g(s).$$

The characteristic equations are

$$\dot{x} = -y, \tag{1a}$$
$$\dot{y} = x, \tag{1b}$$
$$\dot{u} = u. \tag{1c}$$

Differentiating (1a) with respect to t and using (1b) we find $\ddot{x} + x = 0$, which has solution $x(t,s) = s\cos t$, where we used the initial condition. Similarly we find $y(t,x) = s\sin t$. Equation (1c) can be solved directly and gives, after using the initial condition, $u(t,s) = g(s)e^t$.

Since $y/x = \tan t$ and $x^2 + y^2 = s^2$, we can solve for t and s as functions of x and y, finding

$$u(x,y) = g(\sqrt{x^2+y^2})e^{\tan^{-1}\frac{y}{x}}.$$

From $x(t,s) = s\cos t$ and $y(t,x) = s\sin t$, we have that the characteristics are circles centered at the origin. The solution is defined for $x > 0$ since we have chosen the positive square root when solving for s. Indeed, notice that

$$J = \det \begin{bmatrix} \partial_t x & \partial_s x \\ \partial_t y & \partial_s y \end{bmatrix} = \partial_t x \partial_s y - \partial_s x \partial_t y$$
$$= -s\sin t \sin t - (\cos t)s\cos t = -s,$$

So that $J(t,0) = 0$, indicating a potential problem at $s = 0$. (What would happen if we had chosen the negative root?)

(b) We parametrize the initial condition as in (a). The characteristic equations are

$$\dot{x} = 1,$$
$$\dot{y} = 1,$$

$$\dot{u} = u^2.$$

The solutions are $x(t,s) = s + t$, $y(t,s) = t$, and $u(t,s) = \frac{g(s)}{1-tg(s)}$. But $s = x - t = x - y$, hence

$$u(x,y) = \frac{g(x-y)}{1 - yg(x-y)}.$$

The characteristics are straight lines: $y = x - s$. This solution is defined as long as $1 - yg(x-y) \neq 0$. For $g(x) = x^2$, we obtain

$$u(x,y) = \frac{(x-y)^2}{1 - y(x-y)^2}.$$

(c) Parametrize the initial condition as $x_0(s) = s$, $y_0(s) = s + s^2$, $u_0(s) = \sin s$, $s > 0$. The characteristic equations are

$$\dot{x} = 1,$$
$$\dot{y} = 1,$$
$$\dot{u} = 1 - u.$$

We readily find

$$x(t,s) = t + s,\ y(t,s) = t + s + s^2,\ u(t,s) = 1 - (1 - \sin s)e^{-t}.$$

Using the equation for x into the equation for y gives $s = \sqrt{y-x}$, where we chose the positive root according to $x > 0$. Then $t = x - \sqrt{y-x}$, thus

$$u(t,x) = 1 - (1 - \sin\sqrt{y-x})e^{-x+\sqrt{y-x}}.$$

The solution is defined in the region

$$\{(x,y)\,|\,0 < x < y\}$$

The characteristic curves are lines $y = x + s^2$. Notice that the derivatives of u are not defined at $(0,0)$. Computing the Jacobian, we find

$$J(0,s) = 2s,$$

and we see that the transversality conditions fails at $s = 0$ (which corresponds to $(0,0)$). The geometric interpretation of this, discussed in class, can be easily seen here. The characteristic curve for $s = 0$, $y = x$, is tangent to $\Gamma(s)$ at $s = 0$. Thus, the theorem of existence and uniqueness of solutions does not guarantee a solution valid for $x = y = 0$.

Question 2. Derive the system of characteristic equations for the quasilinear equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u).$$

You can follow closely what was done in class for the linear case.

Solution 2. This is essentially as done in class. In class, we had $c(x,y,u) = c(x,y)u + f(x,y)$. But if you look closely at the derivation, we never used this particular form of c. Thus, the same argument as in class, replacing $c(x,y)u + f(x,y)$ by $c(x,y,u)$, works here.

Question 3. Solve

$$uu_x - uu_y = u^2 + (x+y)^2,$$

with initial condition $u(x,0) = 1$. (*Hint:* after writing the characteristic equations, identify an equation satisfied by $x + y$.)

Solution 3. Parametrize the initial condition by $x_0(s) = s$, $y_0(s) = 0$, $u_0(s) = -1$. The characteristic equations are

$$\dot{x} = u, \tag{2a}$$

$$\dot{y} = -u, \tag{2b}$$

$$\dot{u} = u^2 + (x+y)^2, \tag{2c}$$

Adding (2a) and (2b), we obtain

$$\partial_t(x+y) = 0,$$

which, in light of the initial condition, gives

$$x + y = s. \tag{3}$$

Using (3) into (2c) produces $\dot{u} = u^2 + s^2$, which can be integrated to

$$\frac{1}{s}\tan^{-1}\left(\frac{u}{s}\right) = t + g(s).$$

Using the initial condition we find $g(s) = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$, thus

$$u(t,s) = s\tan\left(st + \tan^{-1}\left(\frac{1}{s}\right)\right). \tag{4}$$

Using (4) into (2a) gives

$$\dot{x} = s\tan\left(st + \tan^{-1}\left(\frac{1}{s}\right)\right).$$

Integrating with respect to t and using the initial condition,

$$x(t,s) = -\ln\left|\frac{\cos(st + \tan^{-1}\left(\frac{1}{s}\right))}{\cos\tan^{-1}\left(\frac{1}{s}\right)}\right| + s. \tag{5}$$

Using (3) into the last term of (5) gives

$$y(t,s) = \ln\left|\frac{\cos(st + \tan^{-1}\left(\frac{1}{s}\right))}{\cos\tan^{-1}\left(\frac{1}{s}\right)}\right|. \tag{6}$$

From (6) we get

$$st = \cos^{-1}\left(\frac{se^y}{\sqrt{1+s^2}}\right) - \tan^{-1}\left(\frac{1}{s}\right), \tag{7}$$

where we used the identity $\cos\tan^{-1} z = \frac{1}{\sqrt{1+z^2}}$. Using (7) so replace st and (3) to replace s in (4) finally gives

$$u(x,y) = e^{-y}\sqrt{1 + (x+y)^2 - (x+y)^2 e^{2y}},$$

where we used the identity $\tan\cos^{-1} z = \frac{\sqrt{1-z^2}}{z}$.

Question 4. As we discussed in class, the method of characteristics requires solving a system of ODEs, the characteristic equations. Therefore, it is important to know when the characteristic equations admit solutions and when such solutions are unique. Review your notes/textbook from ODEs and identify important theorems that guarantee when solutions to systems of ODEs exist and are unique. State at least one such theorem. You can consult, for instance:

- Fundamentals of differential equations and boundary value problems, by Nagle, Saff, and Sinder, chapter 13.

- Ordinary differential equations, by Hartman, chapters II and III.

Solution 4. Check the above references.