VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 1

Question 1. Review multivariable calculus, especially the chain rule in several variables.

Question 2. Verify whether the given function is a solution of the given PDE:

(a) $u(x, y) = y \cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x, y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Question 3. For each PDE below, identify the unknown function and state the independent variables. State the order of the PDE. Write the PDE in the form $F(x, u, Du, \ldots, D^m u) = 0$, i.e., identify the function F. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + uu_x = 0$.

(c) $\displaystyle\sum_{ijk} a^{ijk} \partial^3_{ijk} v + v = 0,$

where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x + y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Question 4. Consider a PDE $F(x, u, Du, \ldots, D^m u) = 0$ and let P be the operator associated with it. Prove that the PDE is linear if and only if P is a linear operator.

Question 5. Consider Maxwell's equations:

$$\begin{aligned}
\operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\
\operatorname{div} B &= 0, \\
\frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\
\frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J,
\end{aligned}$$

where div is the divergence and curl is the curl, also written

$$\operatorname{div} f = \nabla \cdot f, \quad \text{and} \quad \operatorname{curl} f = \nabla \times f.$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?