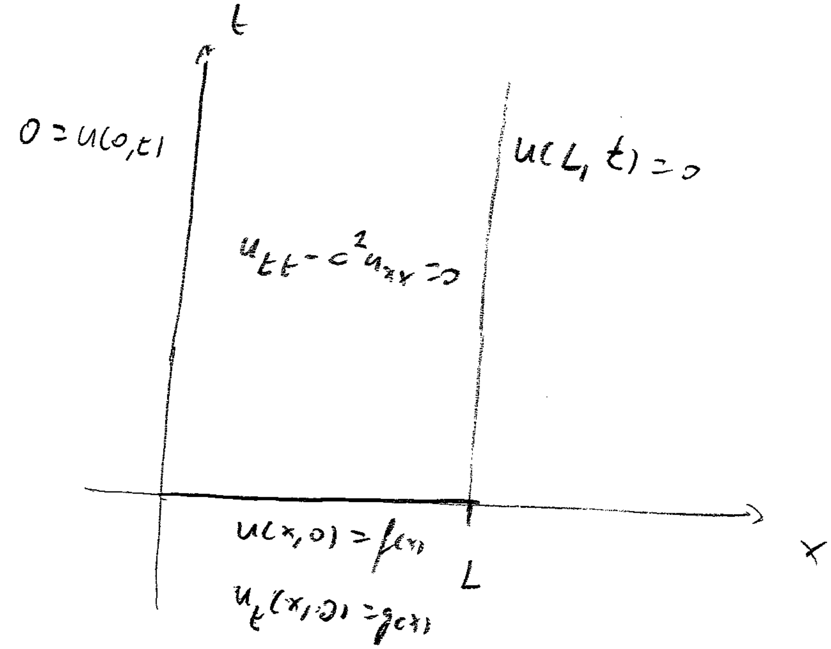
Initial-boundary value problem for the 1d-wave equation: the method of separation of variables.

We now consider the problem

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in} \quad \underset{x}{(0,L)} \times \underset{t}{(0,\infty)}$$

B.C $\{ u(0,t) = u(L,t) = 0 \quad , \quad t \geq 0$

I.C $\begin{cases} u(x,0) = f(x) & , \ 0 \leq x \leq L \\ u_t(x,0) = g(x) & , \ 0 \leq x \leq L \end{cases}$




where $L > 0$, $f$ and $g$ are given functions.

Notice that $f$ and $g$ must satisfy the compatibility conditions:

$$f(0) = f(L) = 0 \quad , \quad g(0) = g(L) = 0$$

In order to solve the problem, we are going to seek solutions that are functions of x and t that are products of a functions only of x and a function only of T. I.e, we suppose that

$$u(x,t) = \mathcal{X}(x)\,T(t)$$

and see if this leads us to a solution. Plugging into the equation:

$$(\mathcal{X}T)_{tt} - c^2(\mathcal{X}T)_{xx} = 0$$

$$\mathcal{X}T'' - c^2 T \mathcal{X}'' = 0 \quad \text{or yet} \quad \frac{\mathcal{X}''}{\mathcal{X}} = \frac{T''}{c^2 T} \quad \text{provided that}$$

$\mathcal{X}$ or $T$ are not zero. But: $\dfrac{\mathcal{X}''}{\mathcal{X}}$ is a function only of $x$ whereas $\dfrac{T''}{c^2T}$ is a function only of $t$. Thus, the only way that equality can hold is if both $\dfrac{\mathcal{X}''}{\mathcal{X}}$ and $\dfrac{T''}{c^2T}$ are equal to a constant, which we denote $\lambda$.

Thus: $\frac{X''}{X} = \lambda$ and $\frac{T''}{c^2 T} = \lambda$. These are now two ODEs for $X$ and $T$ respectively.

<u>Analisys of $X$</u>. The equation reads $X'' - \lambda X = 0$. The characteristic equation is $r^2 - \lambda = 0$, $r = \pm\sqrt{\lambda}$, yielding the general solution

$$X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \quad \text{if} \quad \lambda > 0.$$

$$X(x) = c_1 \cos(\sqrt{-\lambda}\,x) + c_2 \sin(\sqrt{-\lambda}\,x) \quad \text{if} \quad \lambda < 0$$

$$X(x) = c_1 + c_2 x \quad \text{if} \quad \lambda = 0$$

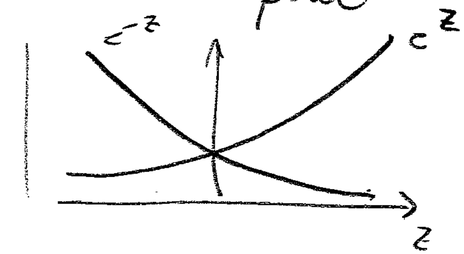
Next, recall that $u(0,t) = 0 = X(0)T(t)$ and $u(L,t) = 0 = X(L)T(t)$, implying $X(0) = X(L) = 0$ since $T \neq 0$.

Let's investigate the three cases above. (Note that we do not know the value of $\lambda$).

If $\lambda > 0$, then we have

$X(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$ and $X(L) = c_1 e^{\sqrt{\lambda} L} + c_2 e^{-\sqrt{\lambda} L} = 0$

$\Rightarrow X(L) = c_1 \left( e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L} \right) = 0$. Since $e^z$ and $e^{-z}$ are equal only when $z = 0$, and $L \neq 0$, $\lambda \neq 0$, this implies $c_1 = 0$. But then $c_2 = 0$ and $X(x) = 0$, giving $u(x,t) = X(x)\,T(t) = 0$. This cannot yield a solution except in the (uninteresting) case $f = g = 0$. Thus we cannot have $\lambda > 0$.




If $\lambda = 0$, then $X(0) = c_1 = 0$ and $X(L) = \overbrace{c_1}^{=0} + c_2 L = 0$, which again gives $u(x,t) = 0$, so we cannot have $\lambda = 0$.

If $\lambda < 0$, then we write $-\lambda = \mu$, with $\mu > 0$, and we can write

$$X(x) = c_1 \cos(\sqrt{\mu}\, x) + c_2 \sin(\sqrt{\mu}\, x).$$

plugging $x=0$ and using $X(0)=0$ gives $c_1\cos 0 + c_2 \sin 0 = c_1 = 0$, thus $X(x) = c_2 \sin(\sqrt{\mu}\, x)$. Now, plug $x=L$ to get

$$X(L) = c_2 \sin(\sqrt{\mu}\, L) = 0,$$ so $c_2 = 0$ or $\sin(\sqrt{\mu}\, L) = 0$. If $c_2 = 0$ then again $u(x,t) = 0$, which cannot give a solution except when $f = g = 0$. Thus, we must have

$$\sin(\sqrt{\mu}\, L) = 0.$$

But this means that $\sqrt{\mu}\, L$ must be a multiple of $\pi$, ie,

$$\sqrt{\mu}\, L = n\pi, \quad n = 1, 2, 3, \ldots$$ (we don't include $n \leq 0$ because $\mu > 0$).

This means that $\mu = \dfrac{n^2\pi^2}{L^2}$ (so $\lambda = -\dfrac{n^2\pi^2}{L^2}$). Therefore, we found what $\lambda$, or equivalently, $\mu$ has to be, except that we can assume infinitely many values according to $n = 1, 2, 3, \ldots$

In other words, the problem $X'' - \lambda X = 0$, $X(0) = X(L) = 0$ admits non-trivial solutions only when $-\lambda = \mu > 0$ (i.e., $\lambda < 0$), in which case it has infinitely many solutions, one for each $n = 1, 2, 3, \ldots$

So we write
$$\boxed{X_n(x) = \sin\left(\frac{n\pi}{L} x\right)}$$

We can now move to the $T$ equation, using that $-\lambda = \mu = \frac{n^2\pi^2}{L^2}$:

$$\frac{T''}{c^2 T} = \lambda \Rightarrow T_n'' + \frac{c^2 n^2 \pi^2}{L^2} T_n = 0,$$ where we write $T_n$ because this is a different equation for each $n$. From ODE theory, we know that the solutions $T_n$ are given by

$$\boxed{T_n(t) = a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)}$$ where $a_n$ and $b_n$ are arbitrary constants.

Since $u = X T$, we see that we have obtained infinitely many functions $u_n$, $n = 1, 2, 3, \ldots$ given by

$$u_n(x,t) = X_n(x)\, T_n(t)$$

$$\boxed{u_n(x,t) = \left( a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right) \right) \sin\left(\frac{n\pi}{L} x\right)}$$

for $n = 1, 2, 3, \ldots$. All these $u_n$'s solve $u_{tt} - c^2 u_{xx} = 0$ and satisfy $u(0,t) = 0 = u(L,t)$. However, we don't know yet that they satisfy the initial condition. Imposing the initial condition will determine the coefficients $a_n$ and $b_n$.

By linearity, any finite sum $\sum_{n=1}^{N} \left( a_n \cos\left( \frac{cn\pi}{L} t \right) + b_n \sin\left( \frac{cn\pi t}{L} \right) \right) \sin\left( \frac{n\pi}{L} x \right)$ is also a solution of the wave equation and it satisfies the boundary conditions. Since this is true for any $N \geq 1$, we are led to consider the series:

$$u(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos\left( \frac{cn\pi}{L} t \right) + b_n \sin\left( \frac{cn\pi t}{L} \right) \right) \sin\left( \frac{n\pi}{L} x \right)$$

This will only be meaningful if the series converges. Let's assume for now not only that the series converges, but also that we can treat it as a finite sum, in the sense that we can differentiate/integrate term by term, change the summation order, etc. We will discuss these points later on.

Since $u(x,0) = f(x)$ we have

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

Differentiating $u$ w.r.t. $t$:

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-a_n \frac{cn\pi}{L}\sin\left(\frac{cn\pi}{L}t\right) + b_n \frac{cn\pi}{L}\cos\left(\frac{cn\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Since $u_t(x,0) = g(x)$:

$$\sum_{n=1}^{\infty} b_n \frac{cn\pi}{L}\sin\left(\frac{n\pi}{L}x\right) = g(x).$$

Recall that $f$ and $g$ are given. The above expressions tell us that $f$ and $g$ can be written as a sum (series) of trigonometric functions, and we want to use this information to find $a_n$ and $b_n$. For this, it is useful to first understand the general question of how to write a given function as a sum (series) of trigonometric functions. This is the theme of <u>Fourier series</u>.