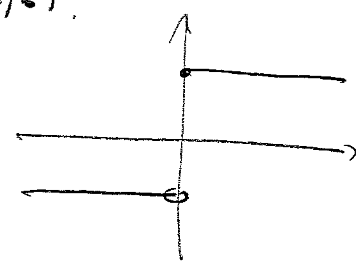
We derived (*) assuming that F and G are C^2. But given formula (*), we can ask if it is valid to consider functions that are not C^2. For instance, take $F(x) = G(x) = |x|$. Then (*) would give

$$u(x,t) = |x+ct| + |x-ct|$$

Since $|x|$ is C^2 except at $x=0$, we see that the above formula fails to give a solution to the wave equation only when $x = \pm ct$, so it seems that $|x+ct| + |x-ct|$ is "almost" a solution. This motivates the following:

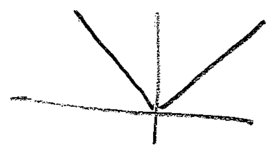
<u>Def</u>. A piecewise C^k function $f : (a,b) \to \mathbb{R}$ is a function that is C^k except at a countable number of isolated points $\{x_i\}_{i=1}^{\infty} \subset (a,b)$.

<u>Ex</u>: $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ is a piecewise smooth (C^∞) function

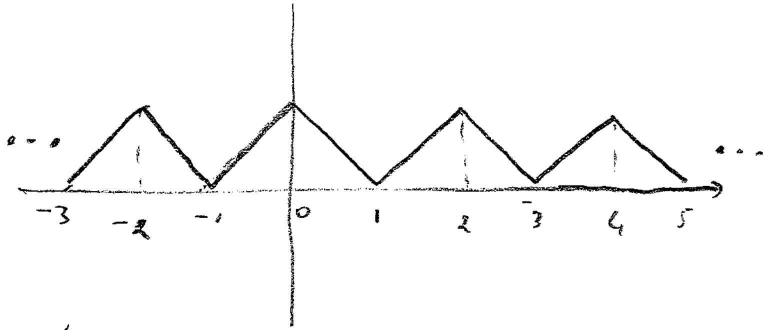


In this example $\{x_i\}_{i=1}^{\infty} = 1$ point $= 0$.

<u>Ex</u>: $f(x) = |x|$ is a piecewise smooth function

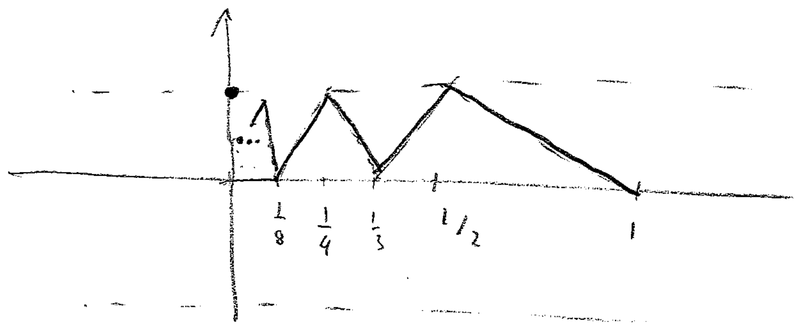


Ex: The function



is a piecewise smooth function. In this case $\{x_i\}_{i=1}^{\infty}$ = integers.

Ex: The function



is not piecewise smooth because in this case $\{x_i\}_{i=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$, and thus $\{0\}$ is not isolated.

Remarks:

- The points $\{x_i\}_{i=1}^{\infty}$ where a piecewise C^k function fails to be C^k are sometimes called the singularities of the function.
- If $f:(a,b) \to \mathbb{R}$ is piecewise C^k and (a,b) is finite, then the number of singularities must be finite. But there can be infinitely many singularities if (a,b) is infinite (ex, $(a,b) = (-\infty, \infty)$, see above example).

Def. Let F and G be piecewise C^2 functions. We call the function

$$u(x,t) = F(x+ct) + G(x-ct)$$

a generalized solution of the wave equation (aka a weak solution of the wave equation).

A generalized solution satisfies the wave equation except at the singularities.

A solution that contains no singularities (i.e., a solution in the "old" sense) is called a classical solution.

Generalized solutions are important, e.g. in the study of shocks. The terminology generalized or weak solution may carry different meanings depending on the context.