In what follows, we shall use the following notation.

Let u be a function defined in a domain $\mathcal{U} \subseteq \mathbb{R}^n$ (possibly $\mathcal{U} = \mathbb{R}^n$). We write $u \in C^k(\mathcal{U})$, or simply $u \in C^k$ when $\mathcal{U}$ is implicitly understood, to mean that all derivatives of u up to order k exist and are continuous in $\mathcal{U}$.

If all derivatives of u of arbitrary order exist, we write $u \in C^\infty$. In the latter case we say that u is smooth.

We also recall the notation $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^3}{\partial x_n^2}$ for the Laplacian. In particular, in 1, 2, and 3 dimensions, we have, respectively,

$\Delta = \partial_x^2$, $\Delta = \partial_x^2 + \partial_y^2$, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$

The wave equation

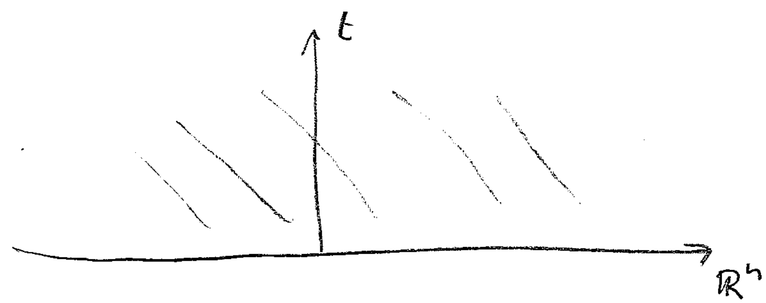
We shall now study the wave equation

$$\partial_t^2 u - c^2 \Delta u = 0$$

for a real valued function $u = u(t, x_1, \dots, x_n)$, $(t, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$. Above, $c > 0$ is a constant that corresponds to the speed of propagation of waves (to be seen).

Typically, in evolution problems, we prescribe initial conditions. We are interested in the Cauchy problem for the wave equation

$$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= 0 \quad \text{in } [0, \infty) \times \mathbb{R}^n \\ u &= g \quad \text{on } \{t = 0\} \times \mathbb{R}^n \\ \frac{\partial u}{\partial t} &= h \quad \text{on } \{t = 0\} \times \mathbb{R}^n \end{aligned}$$



<u>Remark:</u> Cauchy problem = initial value problem.

Case $n=1$

Let's start with the $n=1$ case
$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0,\infty)\times\mathbb{R} \\ u = g & \text{on } \{t=0\}\times\mathbb{R} \\ u = h & \text{on } \{t=0\}\times\mathbb{R} \end{cases}$$

We first verify that any C^2 function $F(x \pm ct)$ satisfies the equation

$u(t,x) = F(x \pm ct)$ $\quad u_t(t,x) = \pm c F'(x \pm ct)$, $u_{tt}(t,x) = c^2 F''(x \pm ct)$

$u_x(t,x) = F'(x \pm ct)$, $u_{xx}(t,x) = F''(x \pm ct)$

So $u_{tt} - c^2 u_{xx} = 0$.

In particular, $u(t,x) = F(x+ct) + G(x-ct)$ is also a solution by linearity, (F, G, C^2 functions). Since $u(0,x) = g(x)$: $F(x) + G(x) = g(x)$. Evaluating $u_t(t,x)$ at $t=0$ yields $u_t(0,x) = cF'(x) - cG'(x) = h(x)$. Integrating:

$F(x) - G(x) = \frac{1}{c}\int_0^x h(s)\,ds + C_1$, $C_1 = F(0) - G(0)$. We thus obtain an algebraic system for F, G:
$$\begin{cases} F(x) + G(x) = g(x) \\ F(x) - G(x) = \frac{1}{c}\int_0^x h(s)\,ds + C_1 \end{cases}$$

We readily find $F(x) = \frac{1}{2} g(x) + \frac{1}{2c} \int_0^x h(s)\,ds + \frac{C}{2}$, $G(x) = \frac{1}{2} g(x) - \frac{1}{2c} \int_0^x h(s)\,ds - \frac{C}{2}$.

Thus, $u(t,x) = F(x+ct) + G(x-ct)$ gives

$$u(t,x) = \frac{1}{2}\left(g(x+ct) + g(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} h(s)\,ds.$$

This formula is known as D'Alemberts formula.

A straightforward calculation gives

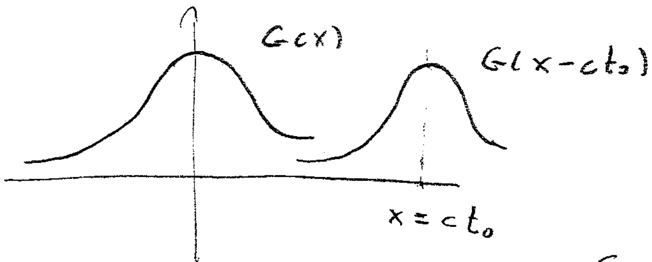
<u>Thm</u> Let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and define u by D'Alembert's formula. Then:

(i) $u \in C^2([0,\infty) \times \mathbb{R})$

(ii) $u_{tt} - c^2 u_{xx} = 0$ in $(0,\infty) \times \mathbb{R}$

(iii) $\displaystyle\lim_{\substack{(t,x) \to (0,x_0) \\ t>0}} u(t,x) = g(x_0)$ and $\displaystyle\lim_{\substack{(t,x) \to (0,x_0) \\ t>0}} u_t(t,x) = h(x_0)$ for each $x_0 \in \mathbb{R}$.

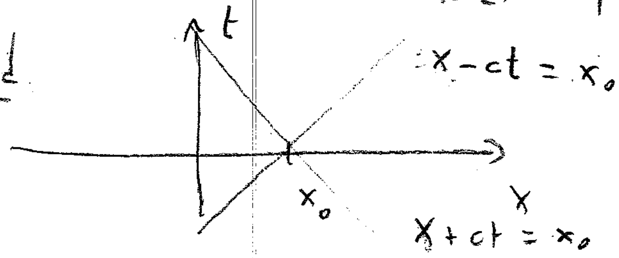
For any fixed $t_0 > 0$, the graph of $G(x - ct_0)$ is the same as that of $G(x)$, except that it is shifted to the right by the distance ct_0. Thus, as t moves, the graph of $G(x-ct)$ moves to the right.

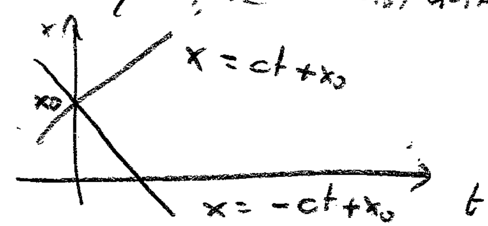


Since the distance travelled by the graph after time t is ct, the graph is moving at speed c.

Similarly, the graph of $F(x+ct)$ moves to the left at speed c.

Since $F(x+ct)$ and $G(x-ct)$ are solutions to the wave equation, we interpret them as waves moving (propagating) at speed c. G is called a <u>forward</u> wave and F a <u>backward</u> wave. Formula (*) above says that any solution to the wave equation in one dimension is a "superposition" sum of a forward and a backward wave. That's why the constant c is the <u>wave speed</u>.

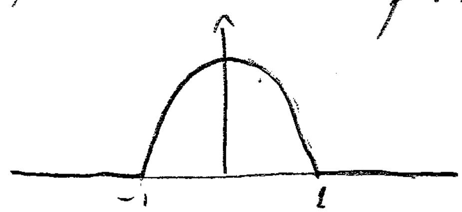




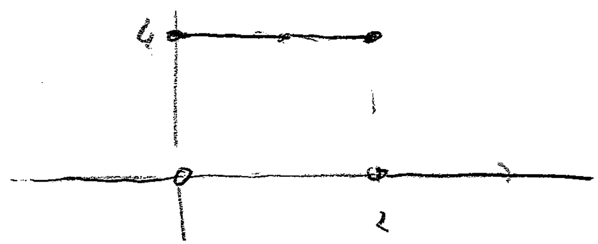
$$\boxed{u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds},$$

Ex: Solve the Cauchy problem for the wave equation with $c = 1$

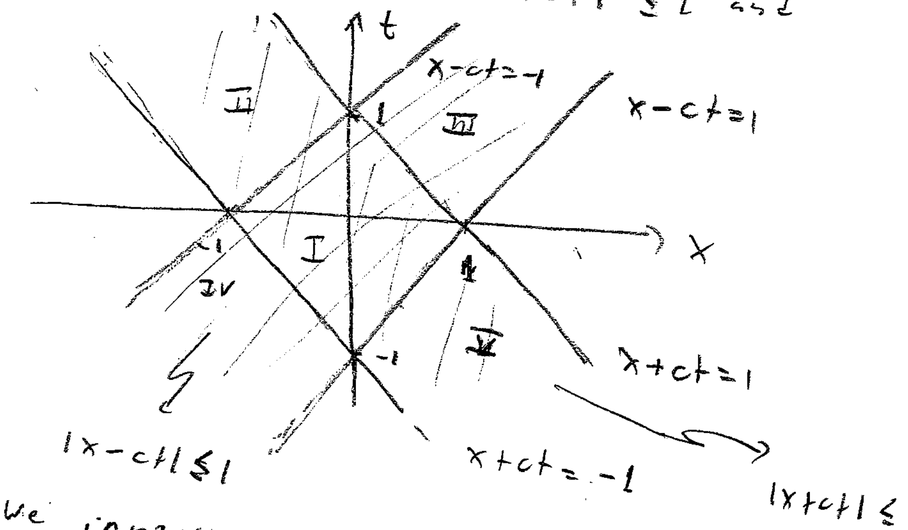
$$f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$



$$g(x) = \begin{cases} 4, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$



Let's use D'Alembert formula. Since the definition of f changes according to $|x| \leq 1$ or $|x| > 1$, we have to consider $|x+ct| \leq 1$ and $|x+ct| > 1$. Similarly for $x-ct$.



$$|x+ct| \leq 1 \Leftrightarrow -1 \leq x+ct \leq 1$$

$$|x-ct| \leq 1 \Leftrightarrow -1 \leq x-ct \leq 1$$

There are five regions, indicated in the picture, where (x,t) satisfies $|x+ct| \leq 1$ or $|x-ct| \leq 1$. We ignore regions IV and V as $t<0$ there.

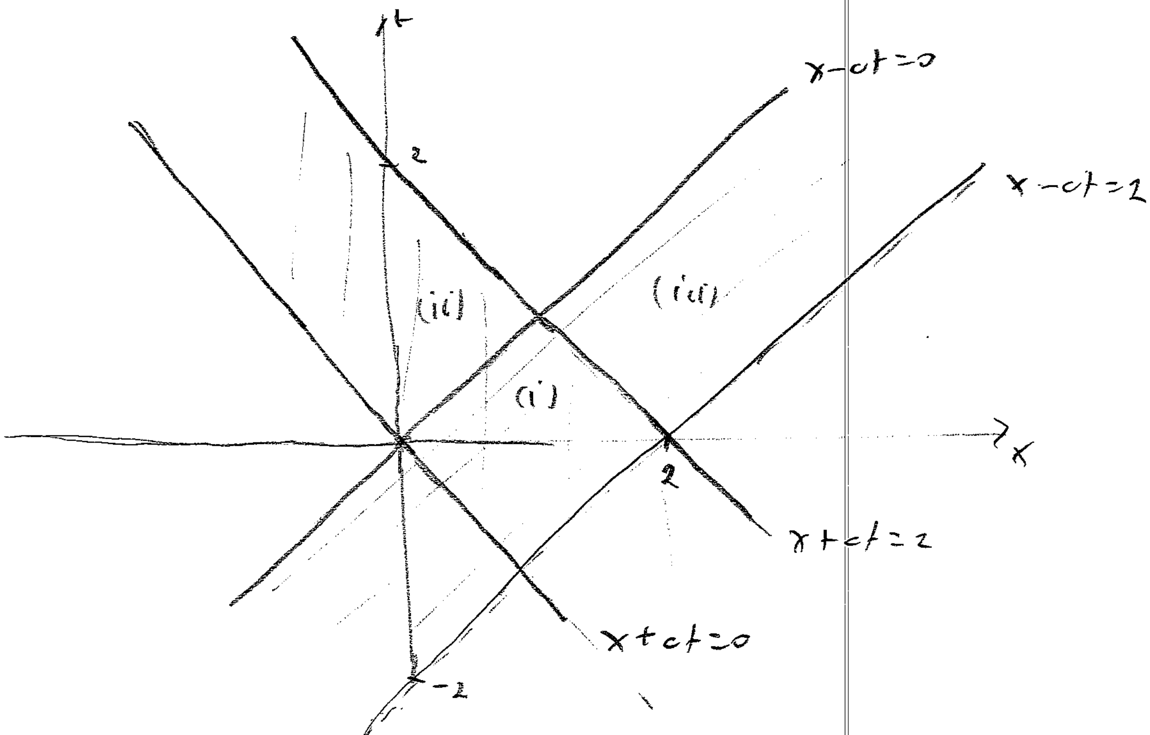
I: $\dfrac{f(x+ct)+f(x-ct)}{2} = \dfrac{1-(x+t)^2 + 1-(x-t)^2}{2} = \dfrac{2-x^2-t^2-2xt-x^2-t^2+2xt}{2} = 1-x^2-t^2$

II: $\dfrac{f(x+ct)+f(x-ct)}{2} = \dfrac{1-(x+t)^2+0}{2} = \dfrac{1-x^2-t^2-2xt}{2}$

III: $\dfrac{f(x+ct)+f(x-ct)}{2} = \dfrac{0+1-(x-t)^2}{2} = \dfrac{1-x^2-t^2+2xt}{2}$

Notice that $t=0$ ~~gives~~ $1-x^2$ in I, and in II and III $t>0$ so we do not test the I.C. there

For g, we consider $0 \le x+ct$, $x+ct \le 2$ and similarly for $x-ct$



Again, we can ignore $t \le 0$.

Notice that for the integral $\int_{x-ct}^{x+ct} g(s)\,ds$, we always have $t \ge 0$.

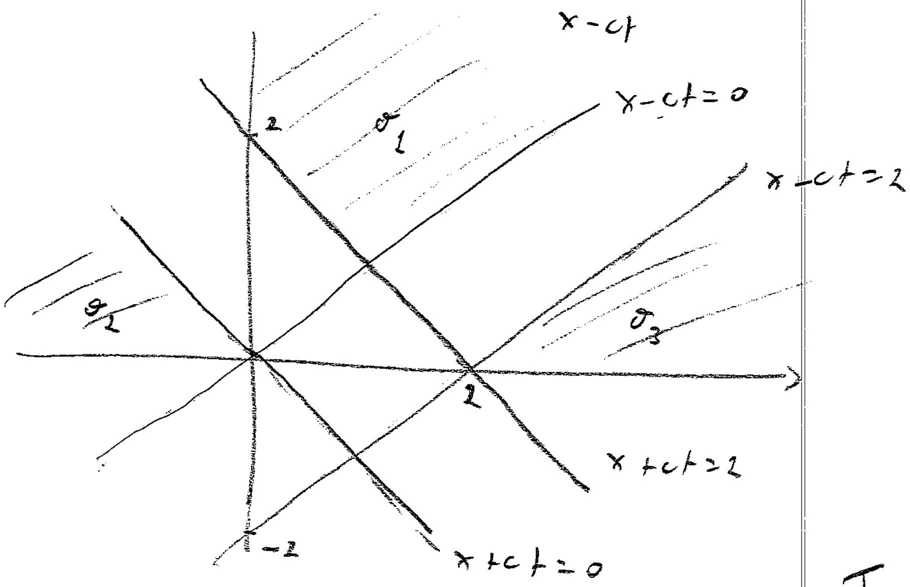
(i) $0 \le x-ct \le x+ct \le 2$, $\frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{x+t} 4\,ds = \frac{1}{2}\left(4(x+t) - 4(x-t)\right) = 4t$

(ii) $x-ct \le 0 \le x+ct \le 2$ $\quad \frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{0}^{x+ct} g(s)\,ds = \frac{1}{2}\int_{0}^{x+t} 4\,ds = 2(x+t)$

(iii) $0 \le x-ct \le 2 \le x+ct$ $\quad \frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{2} g(s)\,ds = \frac{1}{2}\int_{x-t}^{2} 4\,ds = \frac{8-4(x-t)}{2} = 4-2(x-t)$

Notice that we get $\partial_t(\;) = 4$ for $t=0$ in (i).

Next, we analyze $\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$ in the outside regions:



Clearly

$$\int_{x-ct}^{x+ct} g(s)\,ds = 0$$

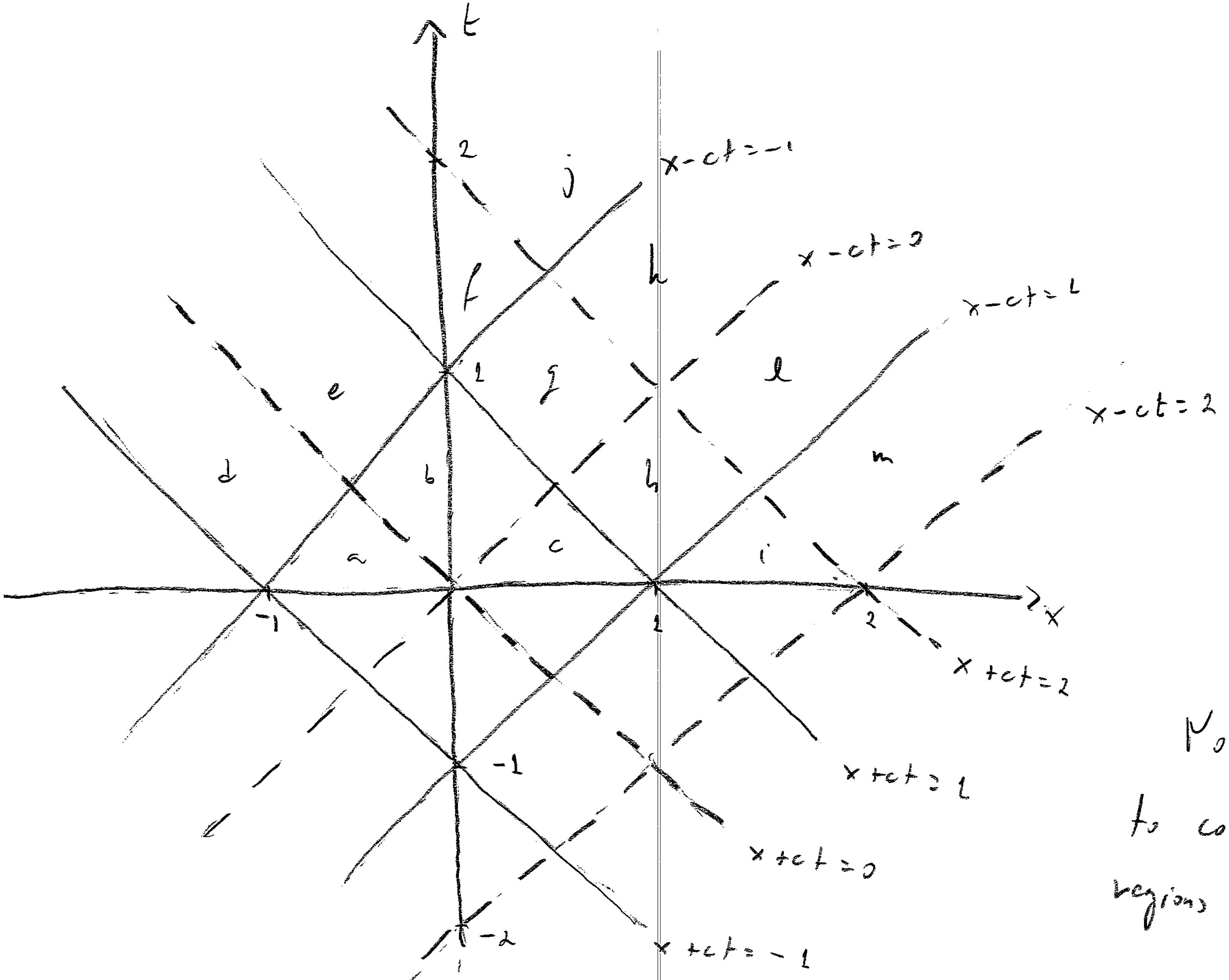
in the regions σ_2 and σ_3.

In the region σ_1, we have $x+ct \geq 2$ and $x-ct \leq 0$, thus,

(σ_1): $x-ct \leq 0 \leq 2 \leq x+ct$, so

$$\frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{0} g(s)\,ds + \frac{1}{2}\int_{0}^{2} g(s)\,ds + \int_{2}^{x+t} g(s)\,ds = 0 + \frac{1}{2}\int_{0}^{2} 4\,ds + 0$$

$$= 4.$$



Now we have to combine the regions for f and g

(a) $-1 \le x+ct$ and $x+ct \le 1$ and $x+ct \le 0$ and $x+ct \le 2$

and

$\underbrace{-1 \le x-ct \text{ and } x-ct \le 1}_{\mathrm{I}}$ and $\underbrace{x-ct \le 0 \text{ and } x-ct \le 2}_{\text{not (i), (ii), or (iii)}}$ $\Rightarrow$ $-1 \le x+ct < 0$ and $-1 \le x-ct < 0$

have to decide on open/closed according to I.C., although here there we observe that $f g$ is continuous.

$$u(x,t) = 1 \underbrace{- x^2 - t^2}_{f \text{ on } \mathrm{I}} + \underbrace{0}_{?}$$

(b) $-1 \le x+ct$ and $x+ct \le 1$ and $0 \le x+ct$ and $x+ct \le 2$

$\underbrace{-1 \le x-ct \text{ and } x-ct \le 1}_{\mathrm{I}}$ and $\underbrace{x-ct \le 0 \text{ and } x-ct \le 2}_{\text{(ii)}}$ $\Rightarrow$ $0 \le x+ct \le 1$ and $-1 \le x-ct < 0$

$$u(x,t) = 1 - x^2 - t^2 + 2(x+t)$$

(c) $-1 \le x+ct$ and $x+ct \le 1$ and $0 \le x+ct$ and $x+ct \le 2$

$\underbrace{-1 \le x-ct \text{ and } x-ct \le 1}_{\mathrm{I}}$ and $\underbrace{0 \le x-ct \text{ and } x-ct \le 2}_{\text{(i)}}$ $\Rightarrow$ $0 \le x+ct \le 1$ and $0 \le x-ct \le 1$

$$u(x,t) = 1 - x^2 - t^2 + 4t$$

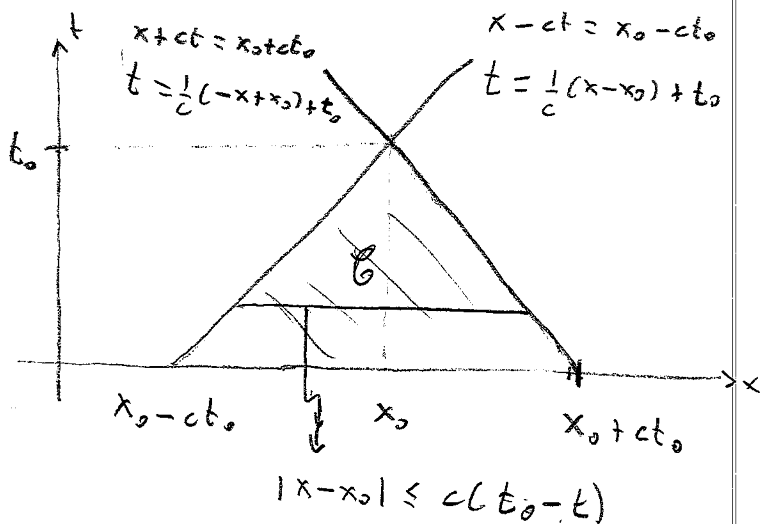
Note that $u(x,0) = 1 - x^2$ in (a) $\cup$ (c), $\partial_t u(x,0) = 4$ in (c), $\partial_t u = 0$ in (a) and that u satisfies the equation in (a), (b), (c).

Proceeding this way we can write the solution

$$u(x,t) = \begin{cases} 1 - x^2 - t^2 & \text{for } -1 \le x+ct < 0 \text{ and } -1 \le x-ct < 0 \quad (a) \\ 1 - x^2 - t^2 + 2(x+t) & \text{for } 0 \le x+ct \le 1 \text{ and } -1 \le x-ct < 0 \quad (b) \\ 1 - x^2 - t^2 + 4t & \text{for } 0 \le x+ct \le 1 \text{ and } 0 \le x-ct \le 1 \quad (c) \\ \vdots & \\ 0 & \text{otherwise} \end{cases}$$

Finite propagation speed Let's see that solutions to the 1d wave equation propagate at speed c.

Let $(x_0, t_0) \in \mathbb{R}^2$, $t_0 > 0$, and consider the lines $x + ct = x_0 + ct_0$ and $x - ct = x_0 - ct_0$ through (x_0, t_0).

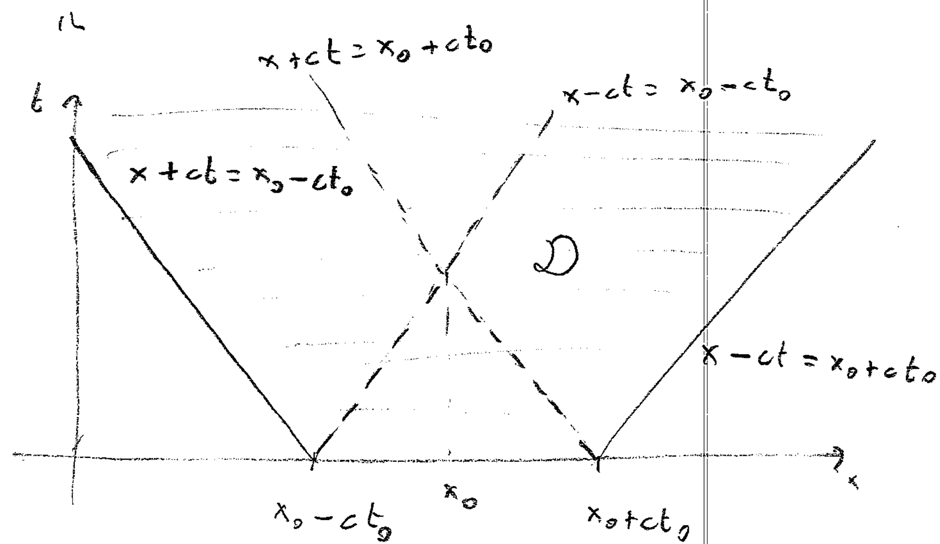


From D'Alembert's formula, we see that if $g = 0 = h$ on $[x_0 - ct_0, x_0 + ct_0]$, then $u(t_0, x_0) = 0$, and in fact $u = 0$ in $\mathcal{C}$, where

$$\mathcal{C} = \{(x,t) \mid 0 \le t \le \tfrac{1}{c}(x - x_0) + t_0\} \cap \{(x,t) \mid 0 \le t \le \tfrac{1}{c}(-x + x_0) + t_0\}$$
$$= \{(x,t) \mid |x - x_0| \le c(t_0 - t),\ t \ge 0\}$$

In particular, if u and $\tilde{u}$ are solutions to the wave equation with initial data g, h and $\tilde{g}, \tilde{h}$, respectively, with $g = \tilde{g}$ and $h = \tilde{h}$ on $[x_0 - ct_0, x_0 + ct_0]$, then $u = \tilde{u}$ in $\mathcal{C}$. This is known as the finite propagation speed property of solutions to the wave equation.

If f and g have support in $[x_0 - ct_0, x_0 + ct_0]$ then $u = 0$ outside $\mathcal{D}$, where



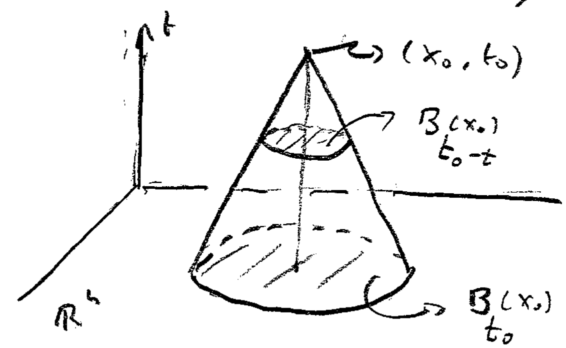
$$\mathcal{D} = \{ (x,t) \mid |x - x_0| \leq c(t_0 + t),\ t \geq 0 \}$$

From the above observations we see that c is in fact the speed of propagation in that $x = \pm ct + \text{constant}$ corresponds to the position of a particle traveling with constant speed c.

Before establishing solutions to the wave equation in $n > 1$, let us investigate some properties of solutions. For simplicity we henceforth set $c = 1$

Finite propagation speed in arbitrary dimensions: Given $(x_0, t_0) \in \mathbb{R}^{n+1}$, $t_0 > 0$, we define cone of dependence $\mathcal{C}(x_0, t_0)$ as

$$\mathcal{C}(x_0, t_0) = \{ (x,t) \in \mathbb{R}^{n+1} \mid 0 \leq t \leq t_0,\ |x - x_0| \leq t_0 - t \}$$



Theo Let $u \in C^2((0,\infty)\times\mathbb{R}^n)$ solve $\partial_t^2 u - \Delta u = 0$, and assume that $u = u_t = 0$ in $B_{t_0}(x_0)$. Then $u = 0$ within $\mathcal{C}(x_0,t_0)$.

Proof: Define the "local energy" $e(t) = \frac{1}{2}\int_{B_{t_0-t}(x_0)} (u_t^2(t,x) + |Du(t,x)|^2)\,dx$, $0 \le t \le t_0$

Then
$$\frac{de(t)}{dt} = \frac{1}{2}\int_{B_{t_0-t}(x_0)} (2u_t u_{tt} + 2\underbrace{Du\cdot Du_t}_{\text{Integrate by parts}})\,dx - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} (u_t^2 + |Du|^2)\,dS$$

$$= \int_{B_{t_0-t}(x_0)} u_t(\overbrace{u_{tt} - \Delta u}^{=0})\,dx + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial \nu} u_t\,dS - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} (u_t^2 + |Du|^2)\,dS = \int_{\partial B_{t_0-t}(x_0)} \left(\frac{\partial u}{\partial \nu} u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2\right) dS$$

The Cauchy-Schwarz inequality gives $\left|\frac{\partial u}{\partial \nu} u_t\right| \le |Du||u_t| \le \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$, so $\frac{de}{dt} \le 0$. Since $e(0) = 0$ and $e(t) \ge 0$, we have $e(t) = 0$ in $\mathcal{C}(x_0,t_0)$, which implies the result. $\square$

Remark: from the above we can derive uniqueness of solutions.