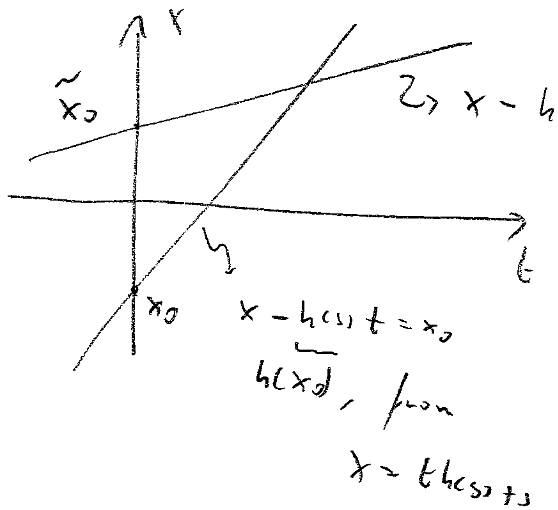
The formula $u(t,x) = h(x - tu(t,x))$ gives u implicitly, but we can double check that it is a solution. Indeed, with $t = \tau$ and writing x in terms of $(\tau, s) = (t,s)$ we have $u(t, x(t,s)) = h(s)$. (where we used $x = \tau h(s) + s = tu + s$)

Taking ∂_t: $u_t(t, x(t,s)) + u_x(t, x(t,s)) \underbrace{\dot{x}(t,s)}_{= u(t, x(t,s))} = 0$

In order to analyze the solutions, let's go back to the characteristic curves: $x(\tau, s) = \tau h(s) + s$; $t(\tau, s) = \tau$, so the (projected) characteristics are $x = th(s) + s$. For each s, the characteristic is a straight line with slope (speed) $h(s)$. In general, different characteristics have different slopes, so they will intersect.



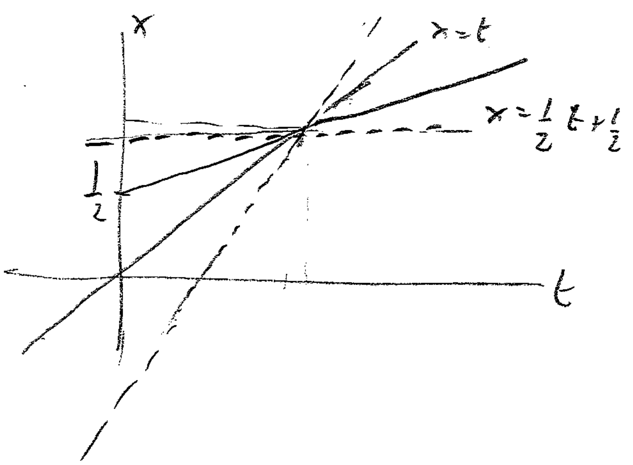
We thus expect something wrong to happen. Indeed, differentiating $u(t,x) = h(x - tu(t,x))$ w.r.t. x:

$u_x(t,x) = h'(x - tu(t,x))(1 - t u_x(t,x))$. Or

$$u_x(t,x) = \frac{h'(x - tu(t,x))}{t h'(x - tu(t,x)) + 1}$$

Or, writing $x - tu = s$ we have $u_x = \dfrac{h'(s)}{1 + t h'(s)}$. Thus, for each s (corresponding to a point on the x-axis, parametrization of the initial condition) there exists a $t_* = t_*(s)$, namely, $t_* = -\dfrac{1}{h'(s)}$, for which $u_x = \infty$. At this moment u is no longer a solution to the PDE. When $u_x = \infty$, we say that the solution <u>blows-up</u>.

To see that the blow-up is indeed due to the intersection of characteristics, let us consider the example $h(s) = 1 - s$. Take the characteristics corresponding to $s = 0$ and $s = \frac{1}{2}$: $x = h(s)t + s \Rightarrow x = t$, $x = \frac{1}{2}t + \frac{1}{2}$.



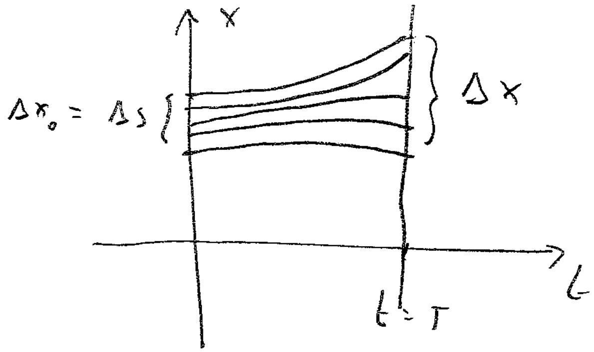
They intersect at $t = 1$. But $h'(s) = -1$, so $t_* = -\dfrac{1}{h'(s)} = 1$, and for $t = 1$ we get $u_x = \infty$.

In fact, all characteristics intersect at $x = 1$:

$x = x_0 + t\,h(x_0) = x_0 + t(1 - x_0) \Rightarrow x(1) = 1$ for any x_0

How can we see that the blow-up is indeed due to intersecting characteristics (as opposed to, say, a coincidence at $t=1$ in the previous example)?

Consider the characteristics $x = t\,h(s) + s$ emanating from an interval of length Δs at $t=0$, and reaching a fixed time $t=T$.



The characteristics leaving $t=0$ reach $t=T$, forming an interval of length Δx at $t=T$. Δx need not to be equal to Δs, but $\Delta x > 0$ unless the interval at $t=T$ is collapsed to a point.

We have $\Delta x \simeq \frac{dx}{ds}\Delta s$ or, infinitesimally, $dx = \frac{dx}{ds}ds$.

When characteristics intersect, $\Delta x \to 0$, ie, $dx = 0$, or $\frac{dx}{ds} = 0$. But computing:

$$\frac{dx}{ds} = t\,h'(s) + 1 = 0 \Rightarrow t = -\frac{1}{h'(s)}, \text{ as before.}$$

The regions where $u_x = \infty$ in Burger's equations are called shock waves, and they correspond to several phenomena of interest

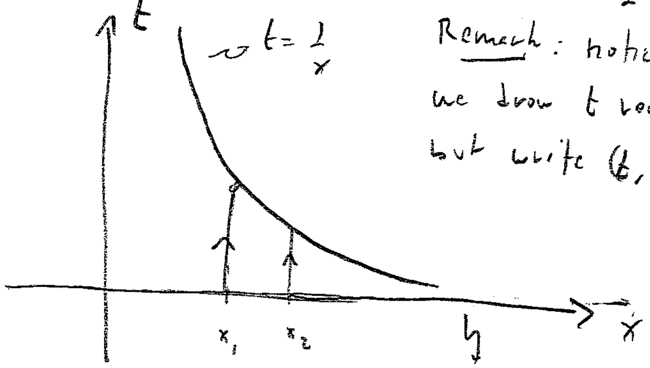
traffic flow, glacier waves, airplanes breaking the sound barrier.

The eq. $u_t + u u_x = 0$ is called Burger's eq (compare to Euler's)

Solutions may exist only locally When a solution $u=u(x,y)$ exists for all (x,y) (or for all (x,y) in a pre-determined range; say, if y represents time we may be interested only on $y \geq 0$), we say that the solution exists *globally*. When a solution exists only on a neighborhood of $\Gamma(s)$ we say that it exists *locally*.

EX: Consider
$$\begin{cases} \dfrac{2}{\pi}\dfrac{1}{x}\dfrac{1}{1+u^2}\, u_t = 1 \\ u(0,x) = 0, \quad x > 0 \end{cases}$$
The characteristic eq. are
$$\begin{cases} \dot{x} = 0 \\ \dot{t} = \dfrac{2}{\pi}\dfrac{1}{x}\dfrac{1}{1+u^2} \\ \dot{u} = 1 \end{cases} \quad \text{with} \quad \begin{aligned} x(0,s) &= s \\ t(0,s) &= 0 \\ u(0,s) &= 0 \end{aligned} \qquad \dot{} = \frac{\partial}{\partial \tau}$$

we find $x = s$, $u = \tau$. Then, plugging into the eq. for t: $\dot{t} = \frac{2}{\pi}\frac{1}{s}\frac{1}{1+\tau^2}$, which has solution $t = \frac{2}{\pi}\frac{1}{s}\tan^{-1}(\tau)$, after using $t(0,s) = 0$. Thus, for each $s = x$, t exists only up to $\frac{1}{x}$, since $\tan^{-1}(\tau) \to \frac{\pi}{2}$ as $\tau \to \infty$. To see what's happening, use $\tau = u$ and solve for u to get $u(t,x) = \tan\left(\frac{\pi x t}{2}\right)$, so u blows-up along $xt = 1$.



$\Gamma(s) = (t(s), x(s))$

Remark: notice that we draw t vertically but write (t,x)

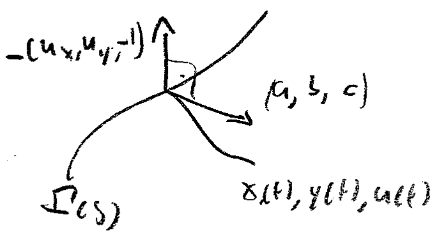
Notice that here the characteristics don't intersect, but they "end" on $t = 1/x$ (with t = time, we would like to have global existence)

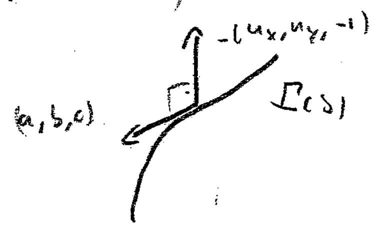
Inversion $(t,s) \mapsto (x,y)$ In the examples we worked out we could always go back to the original variables (x,y), i.e., we could write $t = t(x,y)$, $s = s(x,y)$. But this may not always be the case. Recall that the map $(x,y) = (x(t,s), y(t,s))$ is invertible (so $t = t(x,y)$ and $s = s(x,y)$ are well defined) if the Jacobian $J = \dfrac{\partial(x,y)}{\partial(t,s)}$

$$= \det \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{bmatrix}$$

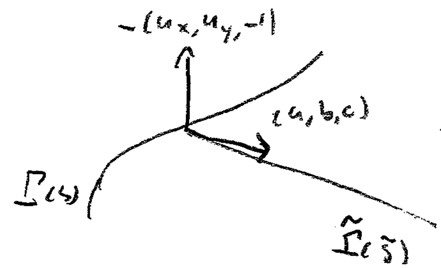
is not zero, $J \neq 0$.

One thing to notice is that while the dependence of the characteristics on t comes from the PDE, the dependence on s comes from the initial condition $\Gamma(s)$. Since the PDE and the initial condition are independent of each other, for a given PDE there always exist initial conditions for which the Jacobian vanishes. These are cases where the characteristic curves don't go "inside" the integral surface:





(a,b,c) doesn't point to the "inside"



(a,b,c) points inside for $\Gamma(s)$, but not for $\tilde{\Gamma}(\tilde{s})$

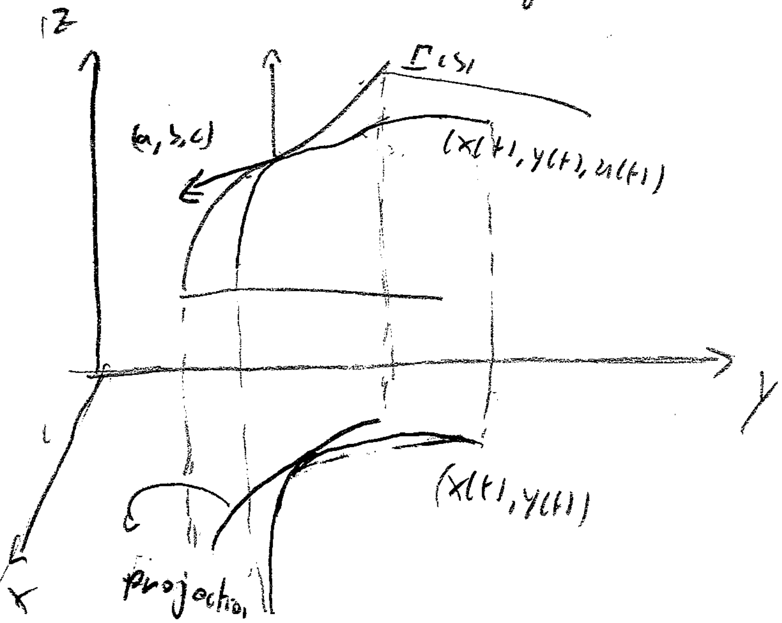
Computing J and evaluating at $t=0$

$J(t,s) = x_t(t,s)\, y_s(t,s) - x_s(t,s)\, y_t(t,s)$ so with $x(0,s) = x_0(s)$, $y(0,s) = y_0(s)$ and

$J(0,s) = a(x_0,y_0)\dfrac{\partial y_0}{\partial s} - b(x_0,y_0)\dfrac{\partial x_0}{\partial s}$, $\quad x_t = a \quad y_t = b$

$\left(a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \right)$

Thus $J(0,s)=0$ if and only if the vectors (a,b) and $\left(\frac{\partial x_0}{\partial s}, \frac{\partial y_0}{\partial s}\right)$ are linearly dependent. $J(0,s)=0$ if the (projected) characteristic curve $(x(t,s), y(t,s))$ at $t=0$ and $\Gamma(s)$ are tangent



of $\Gamma(s) = (x_0(s), y_0(s))$

In general, in order to guarantee a unique solution to a first order PDE with given I.C. we need $J(0,s) \neq 0$ (for all s). $J(0,s) \neq 0$ is called the transversality condition, when it is satisfied $\Gamma(s)$ is said to be non-characteristic and characteristic otherwise.

(I'll give exercise)

Def. Consider

$$(*)\begin{cases} a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \quad \text{with initial condition} \\ \Gamma(s) = (x_0(s),\, y_0(s),\, u_0(s)) \end{cases}$$

We say that the equation and the initial condition satisfy the <u>transversality condition</u> (or that $\Gamma(s)$ is non-characteristic for the PDE) at a point s on Γ if

$$J(0,s) = \underbrace{x_t(0,s)\,y_s(0,s) - y_t(0,s)\,x_s(0,s)}_{} \neq 0$$

$$= \det\begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{bmatrix}\Bigg|_{t=0} = \det\begin{bmatrix} a(x_0,y_0) & \frac{\partial x_0}{\partial s} \\ b(x_0,y_0) & \frac{\partial y_0}{\partial s} \end{bmatrix}, \quad x_0 = x_0(s),\ y_0 = y_0(s)$$

<u>Theorem</u> Consider the Cauchy problem $(*)$, and assume that a, b, and c are <u>smooth functions</u> (i.e., all their partial derivatives of any order exist). Assume that the transversality condition holds for each s in an interval $(s_0 - 2\delta,\, s_0 + 2\delta)$ of the initial curve $\Gamma(s)$. Then, there exists an $\varepsilon > 0$ such that the Cauchy problem $(*)$ has a unique solution in the neighborhood $(t,s) \in (-\varepsilon, \varepsilon) \times (s_0 - \delta,\, s_0 + \delta)$ of the initial curve. If the transversality condition fails for an interval of s values, then $(*)$ has no solution or infinitely many solutions.