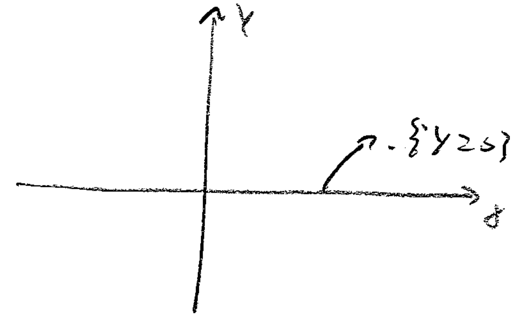


Ex: Solve $u_x + u_y = 2$ with $u(x,0) = x^2$

The characteristic eq. are $(a=1, b=1)$

$$\dot{x} = 1, \quad \dot{y} = 1, \quad \dot{u} = 2.$$

We need to parametrize the initial condition. We can choose $(x, 0) = (s, 0)$, so $x(0,s) = s$, $y(0,s) = 0$, $u(0,s) = s^2$. Solving the eq:

$$x(t,s) = t + f_1(s), \quad y(t,s) = t + f_2(s), \quad u(t,s) = 2t + f_3(s).$$

Using the initial conditions $x(0,s) = s = f_1(s)$, $y(0,s) = 0 = f_2(s)$, $u(0,s) = s^2 = f_3(s)$, thus

$x(t,s) = t+s$, $y(t,s) = t$, $u(t,s) = 2t + s^2$.

$r(t,s) = (x(t,s), y(t,s), u(t,s))$ is a parametric surface corresponding to the graph of u, so it is one form of representing the solution. However, we would like to have $u = u(x,y)$.

So we put $y = t$, $s = x - t = x - y$, so that

$$u(x,y) = 2y + (x-y)^2.$$

Now let's consider the quasilinear case:

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

which we can write as

$$(a,b,c)\cdot(u_x,u_y,-1)=0$$

Thus, as in the linear case, (a,b,c) is tangent to the parametrized surface S given by the graph of (although now the vector (a,b,c) depends not only on (x,y) but also on $u(x,y)$)

Thus we can again write ODE for the curves on S tangent to (a,b,c):

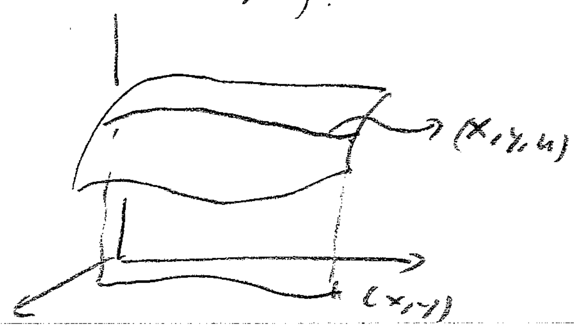
$$\begin{cases} \dot{x}(t,s) = a(x(t,s),y(t,s),u(t,s)) \\ \dot{y}(t,s) = b(x(t,s),y(t,s),u(t,s)) \\ \dot{u}(t,s) = c(x(t,s),y(t,s),u(t,s)) \end{cases}$$

These are again called characteristic eq. and its solutions called characteristic curves. Sometimes we also call characteristic curves their projection onto the xy-plane, i.e. $(x(t,s),y(t,s))$:

with

$$x(0,s)=x_0(s),\quad y(0,s)=y_0(s),\quad u(0,s)=u_0(s)$$

The main difference between the linear and quasi-linear cases is that in the linear case the first two eq. are independent of the third.

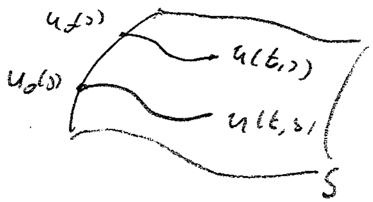


Idea of the method of characteristics

in some cases u does not change at all $\Rightarrow$ u cte along the characteristic

1) Find curves (characteristic curves) along which we can $\overbrace{\text{compute how } u \text{ changes}}$

2) Since we know u at the initial point of those curves (initial condition), we can thus compute u at "any" point on the characteristic curve, thus forming the surface S = graph(u).



Notice how it was important to interpret the equation geometrically to derive the method.

__Terminology__ The problem $\begin{cases} a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \quad (*) \\ x(0,s) = x_0(s),\ y(0,s) = y_0(s),\ u(0,s) = u_0(s) \end{cases}$ is called the __Cauchy problem__ for the quasilinear eq $(*)$

The surface $r(s,t) \equiv (x(t,s), y(t,s), u(t,s))$ is called an __integral surface__ (since the characteristic curves are integral curves).

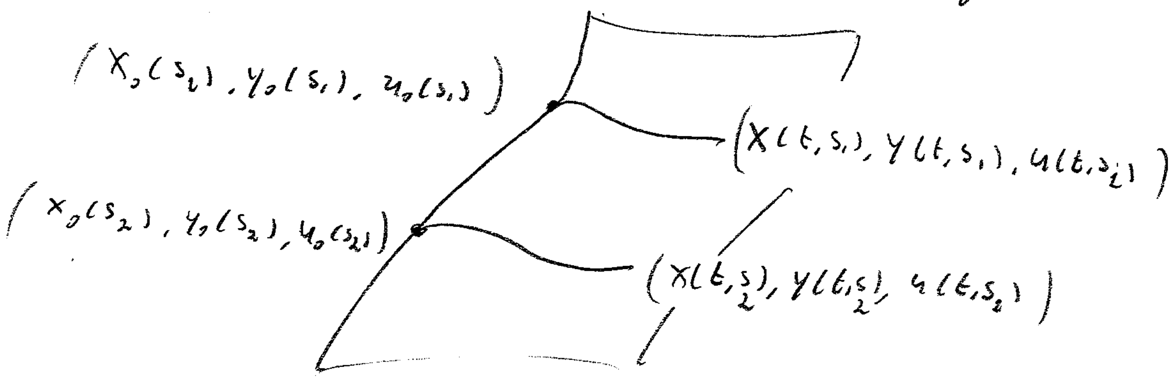
__Remark__. Notice that the characteristic equations can be non-linear even if the PDE is linear (recall ex $3u_x + 2xy^2 u_y = 0$)

__Terminology__: characteristics = characteristic curves.

$\Gamma(s)$ sometimes used for its projection as well. $\Leftarrow$ characteristics (x,y,u) __vs__ projected characteristics (x,y) ($\neq$ characteristics)

What could go wrong?

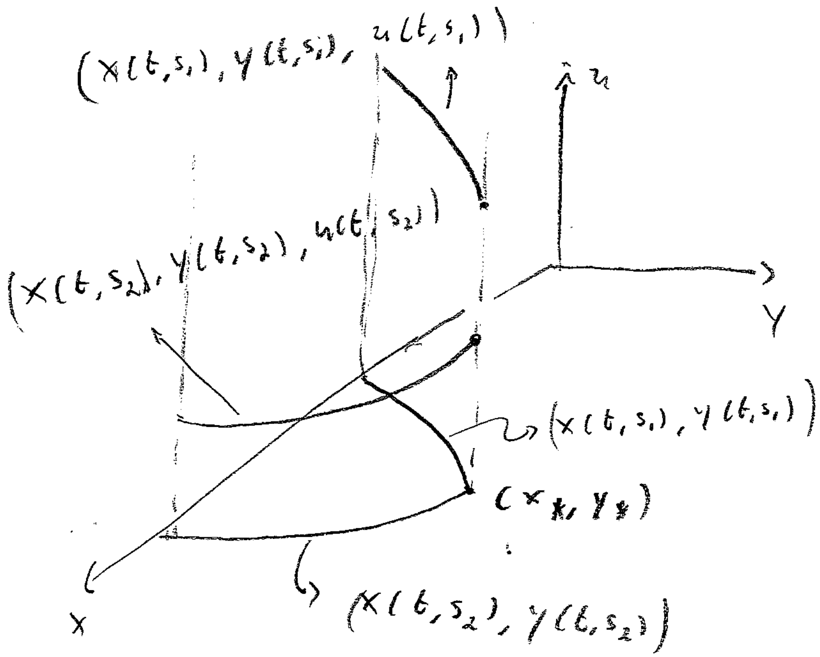
Intersecting characteristics: In the method of characteristics, solutions are constructed by propagating the initial conditions along the characteristic curves.



But what if for some t_*, projected characteristics starting at s_1 and s_2 ($s_1 \neq s_2$) intersect?

Then
$$x(t_*, s_1) = x(t_*, s_2) = x_*$$
$$y(t_*, s_1) = y(t_*, s_2) = y_*$$
but which u do we take for the solution at (x_*, y_*)? $u(t_*, s_1)$ or $u(t_*, s_2)$?

Notice that there is no reason to expect $u(t_*, s_1) = u(t_*, s_2)$



(very important ex)

<u>Ex:</u> Consider $u_t + u u_x = 0$ with $u(x,0) = h(x)$ (so h is a known function)

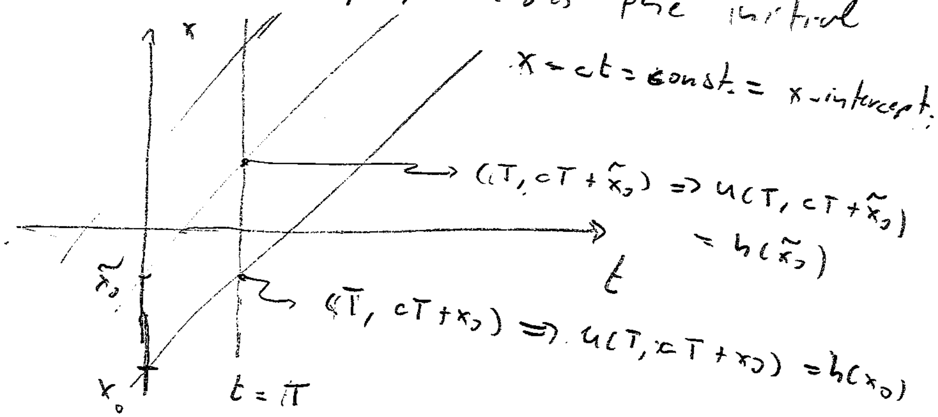
Before analyzing this equation, consider the simpler version $u_t + c u_x = 0$, $c = $ cte.

The characteristic eq. are
$$\begin{cases} \dot{x} = c & x(0,s) = s \\ \dot{t} = 1 & t(0,s) = 0 \\ \dot{u} = 0 & u(0,s) = h(s) \end{cases}$$
where $\dot{} = \frac{\partial}{\partial \tau}$ (since we have label t instead of y)

Then, $(x, t, u) = (s + c\tau, \tau, h(s))$. Eliminating $s = x - c\tau = x - ct$, we find

$u(t,x) = h(x - ct)$. This tells us that the solution simply moves the initial value of u with speed c along the x-axis:

Notice that in this case the (projected) characteristics are all parallel, thus they never intersect.



Back to $u_t + u u_x = 0$, we have
$$\begin{cases} \dot{x} = u \longrightarrow \dot{x} = h(s) \Rightarrow x = \tau h(s) + s \\ \dot{t} = 1 \Rightarrow t = \tau \\ \dot{u} = 0 \Rightarrow u = h(s) \end{cases}$$
Since $h(s) = u(\tau, s)$,
$x = \tau u(\tau, s) + s$

$x(0,s)$, $t(0,s) = 0$

$u(0,s) = h(s)$

Solving for s as before, $s = x - \tau u = x - tu$, so

$u(t,x) = h(s) = h(x - tu)$