The eigenvalue problem for the Laplacian

Consider the problem

$$(*)\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and λ is a constant. We say that λ is an eigenvalue for the Laplacian (with Dirichlet boundary condition) if $(*)$ admits a solution u that is not identically zero. Such an u is called an eigenfunction associated with the eigenvalue λ. The set of all eigenvalues is called the spectrum of the Laplacian. Problem $(*)$ is called the eigenvalue problem for the Laplacian.

We will study properties of eigenvalues and eigenfunctions.

Note the similarity with eigenvalues and eigenvectors in linear algebra:

$Ax = \lambda x$, or $-Ax + \lambda x = 0$, so $-\Delta$ plays the role of the matrix A.

Proposition: Eigenfunctions associated with different eigenvalues are orthogonal to each other

proof: Let u_1 and u_2 be eigenfunctions associated with eigenvalues λ_1 and λ_2, respectively, and assume that $\lambda_1 \neq \lambda_2$. Then, integrating by parts

$$\int_\Omega u_1 \lambda_2 u_2 = -\int_\Omega u_1 \Delta u_2 = \int_\Omega \nabla u_1 \cdot \nabla u_2 - \int_{\partial\Omega} \overset{=0}{\widetilde{u_1}} \frac{\partial u_2}{\partial \nu} = \int_\Omega \nabla u_1 \cdot \nabla u_2 = -\int_\Omega \Delta u_1 \, u_2 + \int_{\partial\Omega} \frac{\partial u_1}{\partial \nu} \overset{=0}{\widetilde{u_2}}$$

$$= -\int_\Omega (-\lambda_1) u_1 u_2$$

Therefore, since λ_1 and λ_2 are constants:

$$(\lambda_2 - \lambda_1) \int_\Omega u_1 u_2 = 0$$

Since $\lambda_2 \neq \lambda_1$, this implies $\int_\Omega u_1 u_2 = 0$, or, recalling the definition of inner product for functions, $\langle u_1, u_2 \rangle = \int_\Omega u_1 u_2 = 0$.

We have been working mostly with real valued functions, but the next proposition shows that even if u can be complex, the eigenvalues are always real.

Proposition The eigenvalues of the Laplacian are always real

Proof. From $\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ we have $\begin{cases} \Delta \bar{u} + \bar{\lambda} \bar{u} = 0 & \text{in } \Omega \\ \bar{u} = 0 & \text{in } \partial\Omega \end{cases}$, where $\bar{u}$ and $\bar{\lambda}$ are the complex conjugate of u and λ, respectively. If λ is not real, then $\lambda \neq \bar{\lambda}$. By the previous proposition, u and $\bar{u}$ have to be orthogonal (note that in the previous proposition it was not assumed that u or λ were real), thus $\int_\Omega u\bar{u} = 0$. But $u\bar{u} = |u|^2$. Since u is not identically zero by definition of eigenfunctions, $\int_\Omega |u|^2 > 0$, hence we have $0 = \int_\Omega |u|^2 > 0$, which is a contradiction. Therefore λ is real. $\blacksquare$

Proposition The eigenvalues are positive.

Proof: Multiply $\Delta u + \lambda u = 0$ and integrate by parts to find:

$$0 = \int_\Omega (u \Delta u + \lambda u^2) = -\int_\Omega \nabla u \cdot \nabla u + \int_{\partial\Omega} \overset{=0}{\tilde{u}} \frac{\partial u}{\partial \nu} + \lambda \int_\Omega u^2 .$$

Since u is not identically zero, $\int_\Omega u^2 > 0$ and we can divide by it, obtaining $\lambda = \dfrac{\int_\Omega |\nabla u|^2}{\int u^2} > 0$, since $\int_\Omega |\nabla u|^2 = 0$ would imply $u = 0$ in view of $u|_{\partial\Omega} = 0$. $\boxtimes$

As in linear algebra, we call the number of linearly independent eigenfunctions associated with an eigenvalue the multiplicity of the eigenvalue. Note, however, that since we are talking about functions and we can have infinitely many linearly independent functions, in principle the multiplicity of an eigenvalue could be ∞. The next proposition says that this does not happen.

<u>Proposition</u> Consider the eigenvalue problem for the Laplacian. Then

(i) The multiplicity of each eigenvalue is finite.

(ii) Let λ be an eigenvalue of multiplicity k, and $u_1, \dots, u_k$ k linearly independent eigenfunctions. Put $E_\lambda = \text{span}\{u_1, \dots, u_k\}$ (i.e, in the language of linear algebra, E_λ is the eigenspace of λ). Then, E_λ admits an orthonormal basis of real eigenfunctions.

(iii) The multiplicity of the smallest eigenvalue (which we know to be positive) is one. Moreover, the corresponding eigenspace is spanned by an eigenfunction u that is positive in Ω, i.e, $u(x) > 0$ for all $x \in \Omega$ (although, of course, $u(x) = 0$ for $x \in \partial\Omega$). (The smallest eigenvalue is well-defined because of iv below.)

(iv) The set of eigenvalues (i.e, the spectrum) forms a discrete sequence $\{\lambda_n\}_{n=1}^\infty$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \dots$ with $\lambda_n \to \infty$. In particular, there are infinitely many eigenvalues.

Remark: Notice that if we only knew that $\{\lambda_n\}_{n=1}^{\infty}$ forms an infinite discrete sequence then we could not claim that there are infinitely many different eigenvalues, since we could have $\lambda_n = \lambda_{n+1} = \lambda_{n+2} = \cdots$ for some n. It's the statement $\lambda_n \to \infty$ that guarantees the existence of infinitely many eigenvalues.

A proof of this proposition is beyond the scope of this course. Using the proposition, we can derive the following consequence.

Since different eigenvalues have orthogonal eigenfunctions, the eigenspace of a given eigenvalue can be spanned by orthonormal eigenfunctions, and there are infinitely many eigenvalues, we conclude that there exists a infinite sequece $\{u_n\}_{n=1}^{\infty}$ of orthonormal eigenfunctions. These functions play a similar role as $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$ in Fourier series.

More precisely, it can be showed that if f is a function defined in Ω satisfying appropriate hypotheses, then f admits a "generalized Fourier series":

$$f(x) = \sum_{n=0}^{\infty} a_n u_n(x), \quad \text{where the } a_n \text{ are constants.}$$

We shall not discuss in detail what the above appropriate hypotheses are, neither the precise meaning of the convergence of the above series. It suffices to say that for most functions appearing in applications, such generalized Fourier series works out. Note that the constants a_n can be computed using $a_n = \langle f, u_n \rangle = \int_\Omega f u$.

<u>Proposition</u> Let $V = \{ v \in C^2(\Omega) \cap C^0(\bar{\Omega}) \mid v \not\equiv 0,\ v|_{\partial\Omega} = 0 \}$.
Then the smallest eigenvalue λ_1 is given by:

$$\lambda_1 = \inf_{v \in V} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}$$

A proof of this proposition is beyond the scope of this course. The above formula is called the Rayleigh-Ritz formula. Note that the infimum is achieved for eigenfunctions of λ_1.