Formal definition of PDEs

Let us first define PDEs for a function of two variables.

Def. A first order PDE for a function $u = u(x,y)$ (the unknown) is an equation that can be written as

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$$

or in simpler form

$$F(x, y, u, u_x, u_y) = 0 \quad (*)$$

for some function F defined on a subset of $\mathbb{R}^5$, i.e., $F : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^5$. A solution to a PDE is a function u that satisfies the equality $(*)$.

Ex: Consider $u_x + u_y = 0$. This can be written as $F(x,y,u,u_x,u_y) = 0$ where $F(p_1, p_2, p_3, p_4, p_5) = p_4 + p_5$.

Ex: Consider $(u_x)^2 + x u_x + y u_y = 0$. This can be written as $F(x,y,u,u_x,u_y) = 0$ where $F(p_1, p_2, p_3, p_4, p_5) = p_4^2 + p_1 p_4 + p_2 p_5$

Remarks on terminology and notation.

Recall that $\mathbb{R}^5 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and more generally $\mathbb{R}^n = \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n \text{ times}}$.

$D \subseteq \mathbb{R}^5$ means that D is a subset of $\mathbb{R}^5$ (possibly $D=\mathbb{R}^5$), and $F: D \to \mathbb{R}$ means that F is a function with domain D and co-domain $\mathbb{R}$ (thus, for each quintuple $(p_1, p_2, p_3, p_4, p_5) \underset{\text{belongs}}{\in} D$, $F(p_1,p_2,p_3,p_4,p_5)$ is a real number, i.e., $F(p_1,p_2,p_3,p_4,p_5) \in \mathbb{R}$).

Notice that when we write a function F for a given (first order) PDE, it is important (i.e, p_1 matches x, p_2 matches y, p_3 matches u, etc.

Above we defined first order PDEs. First order meant that the highest derivative appearing is of order one.

<u>Def</u>. A <u>second order PDE</u> for the a function $u=u(x,y)$ (the unknown) is an equation that can be written as

$$F(x,y,u,u_x,u_y,u_{xx},\overbrace{u_{xy},u_{yx}},u_{yy})=0 \text{ for some function } F: D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}^9$$

for most functions of interest, $u_{xy} = u_{yx}$, so we could have omitted u_{yx}, in which case $D \subseteq \mathbb{R}^8$, but it is convenient to write all derivatives.

Ex: $u_{tt} - c^2 u_{xx} = 0$ can be written as $F(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9)$
where $F(p_1, \dots, p_9) = p_9 - c^2 p_6$ (where we denote y by t and $u = u(x,t)$)

Ex: $u_t - k u_{xx} = 0$ can be written as $F(p_1, \dots, p_8, p_9)$
$F(p_1, \dots, p_8) = p_5 - k p_6$.

We can also define first and second order PDEs for functions of more variables. For example, a first order PDE for a function $u = u(x,y,z)$ is

$$F(x, y, z, u, u_x, u_y, u_z) = 0$$

while a second order PDE for $u(x,y,z)$ is

$$F(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}) = 0$$

We will now give the general definition of a PDE for a function of an arbitrary number of variables and of arbitrary order.

Some notations, terminology, and conventions

We will denote $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}}$. Thus an element $x \in \mathbb{R}^n$ is an ordered n-tuple $x = (x^1, x^2, \dots, x^n)$. Notice that we denote the components of x by superscripts (but sometimes subscripts are also used, $x = (x_1, \dots, x_n)$). We think of elements of $\mathbb{R}^n$ as vectors so that the usual vector operations (addition, multiplication by scalars, etc) are defined, e.g.

$$x + y = (x^1, x^2, \dots, x^n) + (y^1, y^2, \dots, y^n) = (x^1 + y^1, x^2 + y^2, \dots, x^n + y^n)$$

We will not employ any special notation for vectors (such as $\vec{x}$ etc).

When $n = 2$ or $n = 3$, we sometimes use (x, y) and (x, y, z) rather than (x^1, x^2) and (x^1, x^2, x^3), respectively. The value of n is sometimes called the dimension of space ($\mathbb{R}^2$ is two-dimensional, $\mathbb{R}^3$ is three-dimensional)

In many scenarios, we will take x to be an independent variable, so that calculus operations such as differentiation, div, curl (when $n=3$) are defined, e.g.

$\operatorname{div} x = \frac{\partial x^1}{\partial x^1} + \cdots + \frac{\partial x^n}{\partial x^n} = n$. The components x^i, $i = 1, \dots, n$ are also called coordinates and for them calculus operations also apply, e.g., $\frac{\partial x^1}{\partial x^2} = 0$, $\frac{\partial x^3}{\partial x^3} = 1$.

Def. A PDE <u>of order m</u> for a function u of n variables, is an equation

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \frac{\partial^2 u}{\partial x_2^2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}, \text{third order derivatives}, \ldots, \frac{\partial^m u}{\partial x_1^m}, \frac{\partial^m u}{\partial x_1 \underbrace{\partial x_2 \cdots \partial x_2}_{m-1 \text{ times}}}, \ldots, \frac{\partial^m u}{\partial x_n^m}\right) = 0$$

where $F : D \to \mathbb{R}$ with

$$D \subseteq \underbrace{\mathbb{R}^n}_{\substack{\text{variables} \\ x_1, \ldots, x_n}} \times \underbrace{\mathbb{R}}_{\substack{\text{function} \\ u}} \times \underbrace{\mathbb{R}^n}_{\substack{1^{st} \text{ partial} \\ \text{derivatives}}} \times \underbrace{\mathbb{R}^{n^2}}_{\substack{2^{nd} \text{ partial} \\ \text{derivatives}}} \times \underbrace{\mathbb{R}^{n^3}}_{\substack{3^{rd} \text{ partial} \\ \text{derivatives}}} \times \cdots \times \underbrace{\mathbb{R}^{n^{m-1}}}_{\substack{\text{partial derivatives} \\ \text{of order } m-1}} \times \underbrace{\mathbb{R}^{n^m}}_{\substack{\text{partial derivatives} \\ \text{of order } m.}}$$

<u>Notation</u> We denote by $D^k u$ the set of all k^{th} order (partial) derivatives of u.

Remark on terminology

Thus the above can be written as $F(x, u, Du, D^2u, \ldots, D^{m-1}u, D^m u) = 0$

Def. A solution for a PDE $F(x, u, \dots, D^k u) = 0$ is a function that satisfies the equality

Terminology. In $F(x, u, \dots, D^k u) = 0$ we also refer to $u = u(x^1, \dots, x^n)$ as the dependent variable and to $x = (x^1, \dots, x^n)$ as the independent variable.

Linear vs non-linear PDEs

Given a PDE $F(x, u, Du, \dots, D^m u) = 0$ we can always write

$$F(x, u, Du, \dots, D^m u) = F_H(x, u, Du, \dots, D^m u) + f(x)$$

where every term in F_H contains u or its derivatives, and f is a function of x only.

EX $u_x^2 + u_y + xy = 0$, $F(p_1, p_2, p_3, p_4, p_5) = p_4^2 + p_4 + p_1 p_2$

$F_H(p_1, p_2, p_3, p_4, p_5) = p_4^2 + p_5$, $f(p_1, p_2) = p_1 p_2$, or

$F_H(u, u_x, u_y) = u_x^2 + u_y$, $f(x,y) = xy$

EX: $x u_x + y u_y + u - xy = 0$, $F(p_1, p_2, p_3, p_4, p_5) = p_1 p_4 + p_2 p_5 + p_3 - p_1 p_2$

so $F_H(p_1, p_2, p_3, p_4, p_5) = p_1 p_4 + p_2 p_5 + p_3$ and $f(p_1, p_2) = -p_1 p_2$

or $F_H(x, y, u, u_x, u_y) = x u_x + y u_y + u$, and $f(x,y) = -xy$.

Def We call $F_H(x, u, Du, \dots, D^m u)$ the homogeneous part of the PDE $F(x, u, Du, \dots, D^m u) = 0$. The PDE is called homogeneous if $F = F_H$ (so $f = 0$) and non-homogeneous otherwise.

Ex: $x u_x + y u_y + u = 0$ is homogeneous, $u_x^2 + u_y = x$ is non-homogeneous.

Def. A PDE $F(x, u, Du, \dots, D^m u) = 0$ is called linear if F_H can be written as

$$F_H(x, u, Du, \dots, D^m u) = F_0(x, u) + F_1(x, Du) + \dots + F_m(x, D^m u)$$

where each F_i is linear on its second argument, i.e., for any functions u and v and any constants a and b:

$$F_0(x, au + bv) = aF_0(x, u) + b F_0(x, v),$$

$$F_1(x, aDu + bDv) = a F_1(x, Du) + b F_1(x, Dv)$$

$$\vdots$$

$$F_m(x, aD^m u + bD^m v) = a F_m(x, D^m u) + b F_m(x, D^m v).$$

The PDE is called non-linear otherwise.

Remark on notation: The above means, for fixed k, $F_k(x, D^k u) =$ sum of terms linear on each partial derivative of order k, e.g.

$$F_1(x, Du) = F_{11}\left(x, \frac{\partial u}{\partial x_1}\right) + F_{12}\left(x, \frac{\partial u}{\partial x_2}\right) + \dots + F_{1n}\left(x, \frac{\partial u}{\partial x_n}\right)$$

Remark Notice that it follows that

$$F_H(x, au + bv, aDu + bDv, \dots, aD^m u + bD^m v) = a F(x, u, \dots, D^m u) + b F(x, v, \dots, D^m v)$$

if the PDE is linear.

Ex: The PDE $xu_x + yu_y + u - xy = 0$ is a (non-homogeneous) linear PDE

$$F_H(x, u, u_x, u_y) = x^2 u_x + y^2 u_y + u$$

$$F_H(x, \underbrace{au+bv}_{\substack{\text{replace}\\ u \text{ by}\\ au+bv}}, \underbrace{au_x+bv_x}_{\substack{\text{replace}\\ u_x \text{ by}\\ au_x+bv_x}}, \underbrace{au_y+bv_y}_{\substack{\text{replace}\\ u_y \text{ by}\\ au_y+bv_y}}) = \overbrace{x^2(au_x+bv_x)}^{F_x(x,y,au_x+bv_x)} + \overbrace{y^2(au_y+bv_y)}^{F_y(x,y,au_y+bv_y)} + \overbrace{au+bv}^{F_0(x,y,au+bv)}$$

$$= ax^2u_x + bx^2v_x + ay^2u_y + by^2v_y + au + bv$$

$$= a(x^2u_x + y^2u_y + u) + b(x^2v_x + y^2v_y + v)$$

$$= a\,F_H(x, u, u_x, u_y) + b\,F_H(x, v, v_x, v_y)$$

Ex: The PDE $u_x^2 + u_y = 0$ is a (homogeneous) non-linear PDE

$$F_H(x, u, u_x, u_y) = u_x^2 + u_y$$

$$F_H(x, au+bv, au_x+bv_x, au_y+bv_y) = (au_x+bv_x)^2 + au_y + bv_y = a^2u_x^2 + b^2v_x^2 + ab\,u_xu_y + au_y + bv_y$$

$$\neq a(u_x^2 + u_y) + b(v_x^2 + v_y)$$