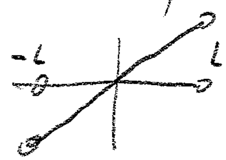
Theorem Let f be a continuous function on $[-L, L]$ and assume that

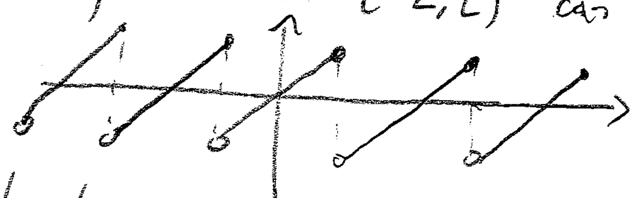
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Then, for any $x \in [-L, L]$, we have

$$\int_{-L}^{x} f(t)\, dt = \int_{-L}^{x} \frac{a_0}{2}\, dt + \sum_{n=1}^{\infty} \left(a_n \int_{-L}^{x} \cos\left(\frac{n\pi t}{L}\right) dt + b_n \int_{-L}^{x} \sin\left(\frac{n\pi t}{L}\right) dt \right)$$

The case of periodic functions Suppose that f is defined on $\mathbb{R}$ and has period $2L$. ($f(x+2L) = f(x)$ for all x). Thus, all information about f is determined by its values on $[-L, L]$. We can then define a Fourier series for f and the previous results are immediately adapted to this case. In fact, any function on $(-L, L)$ can be extended to a periodic function:





Fourier series are often used to study periodic functions. They also appear in several applications (e.g, signal processing)

Relation between series on $[-L, L]$ and $[0, L]$

As we saw, in separation of variables, series of the form $\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ and $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ need to be considered on $[0, L]$. These series are related to Fourier series on $[-L, L]$ as follows. Consider first the case

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ on $[0, L]$. We extend f to an even function on $[-L, L]$ by

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0 \end{cases}$$

We can now compute the Fourier series of $\tilde{f}$ on $[-L, L]$. But because $\tilde{f}$ is an even function, we have

$$a_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

and $b_n = 0$, which is what we had for f on $[0, L]$. Since $\tilde{f}(x) = f(x)$ for $0 \le x \le L$, the Fourier series defined on $[-L, L]$ agrees with the one we had defined on $[0, L]$ for each $x \in [0, L]$.

Similarly, if $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ on $[0,L]$, we can extend f to an odd function on $[-L,L]$ by $\tilde{f}(x) = \begin{cases} f(x), & 0 \leq x \leq L \\ -f(-x), & -L \leq x < 0 \end{cases}$. Computing the Fourier series on $[-L,L]$ and using that $\tilde{f}$ is an odd function, we find $a_n = 0$ and

$$b_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

so again the results on $[-L,L]$ and on $[0,L]$ agree for $x \in [0,L]$.

Convergence of solutions to the wave equation

We haven't yet showed that the found solution

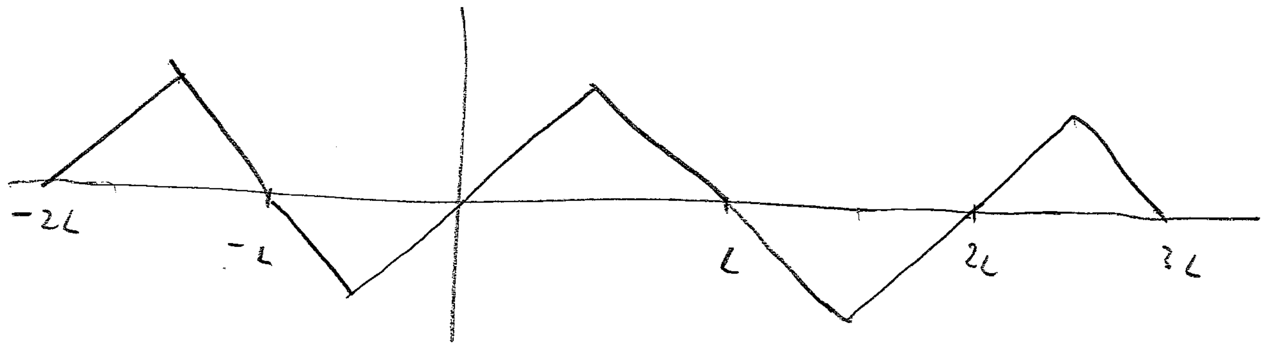
$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$, $b_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$ $\left(\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array} \right)$

indeed provides a solution (classical or generalized) to the problem. There are different ways to do this, depending on the properties of f and g. But it should come as no surprise that we need to use results about convergence of Fourier series.

Let us assume that f and g are piecewise C^1 functions and continuous functions.

Recall that f and g are defined on $[0, L]$ and that $f(0) = g(0) = 0$. (because of compatibility conditions). We can thus make an odd extension of f and g to odd $2L$-periodic functions defined on $\mathbb{R}$. Furthermore, since $f(L) = g(L) = 0$ (again by compatibility conditions) we have $f(-L) = g(-L) = 0$



Denote by $\tilde{f}$ and $\tilde{g}$ the odd extensions of f and g, respectively.

Consider the problem

$$
\begin{cases}
\tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{in } (-\infty, \infty) \times (0, \infty) \\
\tilde{u}(x, 0) = \tilde{f}(x) \\
\tilde{u}_t(x, 0) = \tilde{g}(x)
\end{cases}
$$

The solution to this problem is given by D'Alembert's formula:

$$\tilde{u}(x,t) = \frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(y)\,dy$$

Using our convergence result for Fourier series (adapted to periodic functions, as discussed) we have

$$\tilde{f}(y) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}\left(\tilde{a}_n \cos\left(\frac{n\pi y}{L}\right) + \tilde{b}_n \sin\left(\frac{n\pi y}{L}\right)\right)$$

But since $\tilde{f}$ is odd: $\tilde{a}_n = 0$, $\tilde{b}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(y)\sin\left(\frac{n\pi y}{L}\right)dy = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi y}{L}\right)dy$ (*)

Thus:

$$\tilde{f}(y) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{n\pi y}{L}\right)$$ with $\tilde{b}_n$ given by (*). We can thus compute

$$\frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\tilde{b}_n \sin\left(\frac{n\pi(x+ct)}{L}\right) + b_n \sin\left(\frac{n\pi(x-ct)}{L}\right)\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \tilde{b}_n \left(\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + \cancel{\sin\frac{n\pi ct}{L}\cos\frac{n\pi x}{L}} + \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} - \cancel{\sin\frac{n\pi ct}{L}\cos\frac{n\pi x}{L}} \right)$$

$$= \sum_{n=1}^{\infty} \tilde{b}_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L},$$ where $\tilde{b}_n$ is given by (*)

Similarly, we have $\tilde{g}(y) = \frac{\tilde{A}_0}{2} + \sum_{n=1}^{\infty}\left(\tilde{A}_n\cos\left(\frac{n\pi y}{L}\right) + \tilde{B}_n\sin\left(\frac{n\pi y}{L}\right)\right)$. Again, since $\tilde{g}$ is odd: $\tilde{A}_n = 0$, $\tilde{B}_n = \frac{1}{L}\int_{-L}^{L} g(y)\sin\left(\frac{n\pi y}{L}\right)dy = \frac{2}{L}\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)dy$ (**)

Under our assumptions on g it is valid to integrate the Fourier series term-by-term:

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty}\tilde{B}_n\sin\left(\frac{n\pi y}{L}\right)dy = \frac{1}{2c}\sum_{n=1}^{\infty}\tilde{B}_n\int_{x-ct}^{x+ct}\sin\left(\frac{n\pi y}{L}\right)dy = \frac{1}{2c}\sum_{n=1}^{\infty}-\tilde{B}_n\frac{L}{n\pi}\cos\left(\frac{n\pi y}{L}\right)\Big|_{x-ct}^{x+ct}$$

$$= -\frac{1}{2c}\sum_{n=1}^{\infty}\frac{\tilde{B}_n L}{n\pi}\left(\cos\left(\frac{n\pi(x+ct)}{L}\right) - \cos\left(\frac{n\pi(x-ct)}{L}\right)\right) = -\frac{1}{2c}\sum_{n=1}^{\infty}\frac{\tilde{B}_n L}{n\pi}\left(\cancel{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)}\right.$$

$$\left. - \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) - \cancel{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)} - \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{\tilde{B}_n L}{n\pi c} \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

Hence:

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} \underbrace{\tilde{b}_n}_{\text{call this } a_n} \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \underbrace{\frac{B_n L}{n\pi c}}_{\text{call this } b_n} \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

where from (*) and (**) we have $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$, $b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

But this is exactly the series solution $u(x,t)$, so $\tilde{u}(x,t) = u(x,t)$ for $0 \le x \le L$ and we conclude that the series for u converges since that for $\tilde{u}$ does. Furthermore, $\tilde{u}$ is odd and satisfies $\tilde{u}(0,t) = \tilde{u}(L,t) = 0$, so $\tilde{u}$ and thus u satisfies the boundary conditions. We conclude that the formal solution u is an actual solution.

Remark: in practice, for "nice" initial data, we expect formal solutions to be actual solutions.