Formal Aspects of Fourier series

We will be working on $[-L, L]$ rather than $[0, L]$. The connection to the Fourier series of sines and cosines studied earlier (defined on $[0, L]$) will be made later on.

We will make use of the following:

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L}\, dx = 0, \quad \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \neq 0 \\ 2L & m = n = 0 \end{cases},$$

where m and n are integers.

In general, we say that two functions f, g are <u>orthogonal</u> if $\langle f, g \rangle = 0$. The above equalities thus say that sin and cos $\frac{n\pi x}{L}$ are always orthogonal when $n \neq m$, and the interval is $[-L, L]$

Let f be defined on $[-L,L]$. Our goal is to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right), \quad a_n, b_n \text{ real numbers,}$$

We write the coefficient a_0 separately and differently because of the factor 2 in $\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle$ for $n=0$.

Using the orthogonality of sin and cos just stated, we can immediately compute the coefficients as we did before. They are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3, \ldots \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots$$

This, of course, assumes that the series converges and the limit is f. We don't know if that is the case, but we are led to the following.

Def. Let f be a piece-wise continuous function defined on $(-L, L)$.
(or $[-L, L]$)
The Fourier series of f, denoted F.S.$\{f\}$, or F.S.$\{f\}(x)$, is the series

$$F.S.\{f\}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where a_n and b_n are given by the above formulas, and are called Fourier coefficients

Important remark: The Fourier series F.S.$\{f\}$ is a series constructed out of f, but we are not claiming that $f = F.S.\{f\}$. In fact, we are not even claiming that F.S.$\{f\}$ converges (Although we do want to find conditions that guarantee that F.S.$\{f\}$ converges and $f = F.S.\{f\}$.)

Ex: Find the Fourier series of $f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$

We compute: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$ since f is odd and cos even.
(except for $x=0$)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \underset{\sin \text{ is odd}}{=} \frac{2}{\pi}\int_0^{\pi} \overbrace{f(x)}^{=1}\sin(nx)\,dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd.} \end{cases}$$

Thus

$$F.S\{f\}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\sin(nx)$$

$$= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right)$$

Remark:

$f(0) = 1$ but

$F.S.\{f\}(0) = 0$

<u>Ex:</u> Find the Fourier series for $f(x) = |x|$, $-1 \le x \le 1$.

Compute $b_n = \displaystyle\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = 0$ since f is even

$$a_0 = \int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = 1.$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right), \quad n = 1, 2, \ldots$$

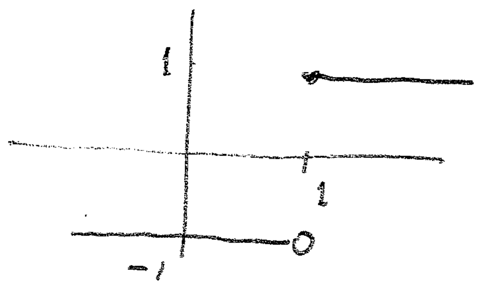
Thus

$$F.S\{f\}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(25\pi x) + \ldots\right)$$

Convergence of Fourier series

<u>Notation</u> We denote by $f(x^+)$ and $f(x^-)$ the right and left values of f at x, defined by $f(x^+) = \lim_{h \to 0^+} f(x+h)$, and $f(x^-) = \lim_{h \to 0^-} f(x+h)$. We have $f(x^+) = f(x^-) = f(x)$ when f is continuous, but $f(x^+)$ and $f(x^-)$ can be different otherwise



$f(1^+) = 1, \quad f(1^-) = -1.$

We will now state results about convergence, differentiation, and integration of Fourier series.

Theorem Let f be a piecewise C^1 function defined on $[-L,L]$. Then, for any $x \in (-L,L)$:

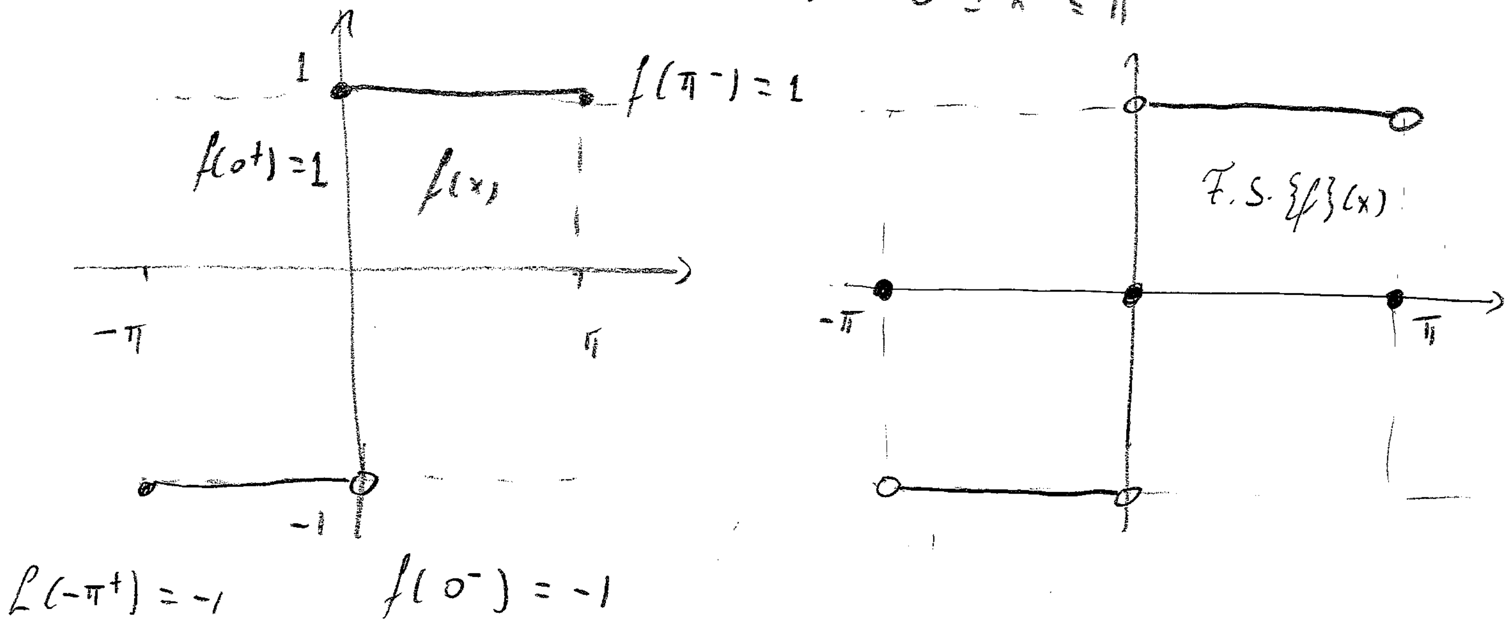
$$\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)}_{= F.S.\{f\}.} = \frac{1}{2}\left(f(x^+) + f(x^-)\right),$$

where a_n and b_n are as before. For $x = \pm L$, the series converges to $\frac{1}{2}\left(f(-L^+) + f(L^-)\right)$

Therefore, if f is piecewise C^1, we have $F.S.\{f\}(x) = f(x)$ if f is continuous at x and $F.S.\{f\}(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$ otherwise. In particular, if f is piecewise C^1 and continuous, we have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

Ex: We graph $f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases}$ and $F.S.\{f\}(x)$ below.



Note the difference between f and $F.S.\{f\}$ at the points of discontinuity and at the endpoints.

Ex: Since $|x|$ is continuous, we have

$$|x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x)$$

Next, we consider differentiation and integration of Fourier series term-by-term, in the spirit of what we did when we solved the wave equation.

Theorem Let f be continuous on $[-L,L]$. Suppose that $f(-L)=f(L)$, and that f is piecewise C^2. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. I.e., if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

then

$$f'(x) = \sum_{n=1}^{\infty}\left(a_n \underbrace{\left(\cos\left(\frac{n\pi x}{L}\right)\right)'}_{=-\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)} + b_n \underbrace{\left(\sin\left(\frac{n\pi x}{L}\right)\right)'}_{=\frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)}\right)$$

We now state a similar result for integration.