Intro to PDEs

Recall that an Ordinary Differential Equation (ODE) is an equation that relates a function and its derivatives. For example, if y is a function of x, $y = y(x)$ (remark on notation) (so y is a dependent variable and x an independent variable), the equation:

$$\frac{d^2y}{dx^2} + y = 0 \qquad (*)$$

is a ODE. It can also be written as $y'' + y = 0$. In an ODE, the <u>function is the unknown</u>, so the question to be answered in this example is: can we find a <u>function</u> y that <u>satisfies</u> $(*)$ (ie, find y such that when y is plugged into $*$) an equality is obtained)?

In this case, the answer is yes. eg $y(x) = \cos x$, then $\frac{d^2y}{dx} + y = -\cos x + \cos x = 0$ ✓. But also $\tilde{y}(x) = \sin x$: $\frac{d^2\tilde{y}}{dx} + \tilde{y} = -\sin x + \sin x = 0$

Compare to an algebraic equation, e.g., $x^2 - 1 = 0$: can we find a number x such that the equation is satisfied.

Notice that, as for algebraic equations, ODEs can admit multiple (more than one) solutions.

If our function is a function of more than one variable, e.g. $z = z(x,y)$, then we need to consider partial derivatives: $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

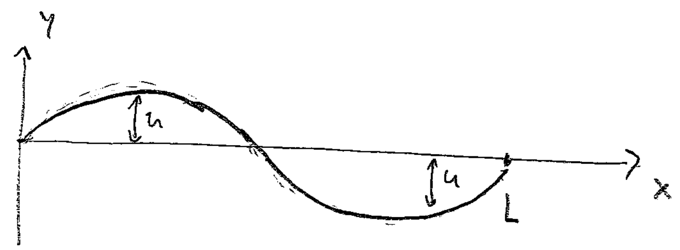
A Partial Differential Equation (PDE) is an equation that relates a function (the unknown) and its partial derivatives.

Ex:
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + z = 0 \quad (**)$$
In this example, the question to be answered is: can we find a function z (of x and y) that satisfies $(**)$? A function $z = z(x,y)$ (remark on notation) that satisfies $(**)$ is called a solution to $(**)$.

Remark: It's always easy to verify if a given function is a solution to a PDE: just plug it in an see if equality is satisfied.

Let's see other examples of PDEs.

Consider a string of length L with both ends attached. Suppose the string can vibrate with an amplitude u. (no motion in the horizontal direction)



u changes over space (x) and time (t), so u is a function of t and x:

$$u = u(t,x)$$

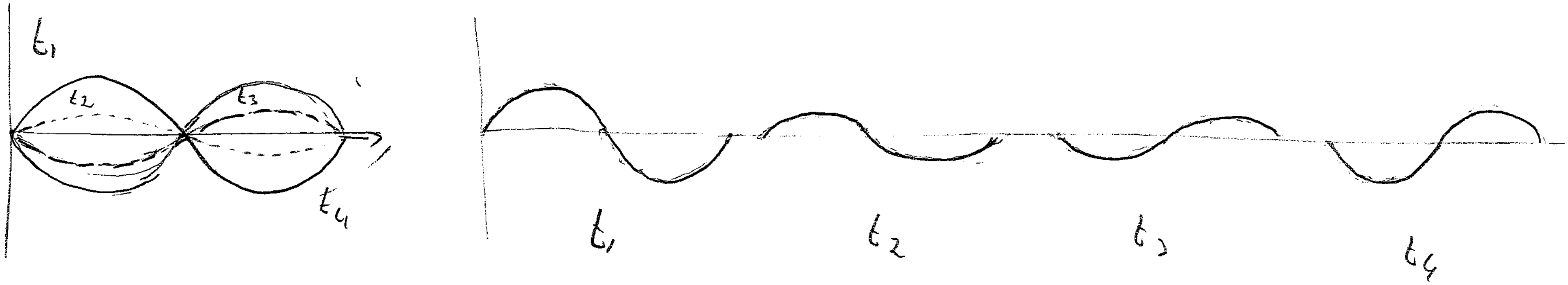
The dynamics (behavior over space and time) of the amplitude u is described by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Above, c is a constant that depends on the nature of the string. If the string has constant density ρ (mass/length) and constant tension T (force), then

$c = \sqrt{\frac{T}{\rho}}$. Notice that $c = \sqrt{\frac{N}{kg/m}} = \sqrt{\frac{N\,m}{kg}} = \sqrt{\frac{kg \cdot m/s^2 \cdot m}{kg}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}$ = units of velocity

We will see later on that c corresponds exactly to the speed of waves propagating along the string.



imagine water waves moving on the ocean surface.

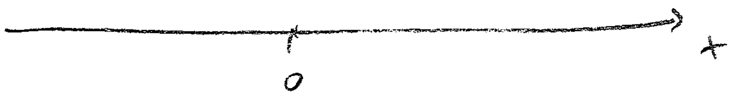
Notice that in this example, a solution u must satisfy.

$$u(t,0) = 0 \quad \text{and} \quad u(t,L) = 0$$

These are extra conditions that must be added to the PDE, and they are called <u>boundary conditions</u> (B.C.). A PDE + B.C is known as a <u>boundary value problem</u>

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(t,0) = 0 = u(t,L) \end{cases}$$

As another example, consider an infinitely long rod (idealization), which we identify with the x-axis:



Let u denote the temperature on the rod. u can vary over space and time, so

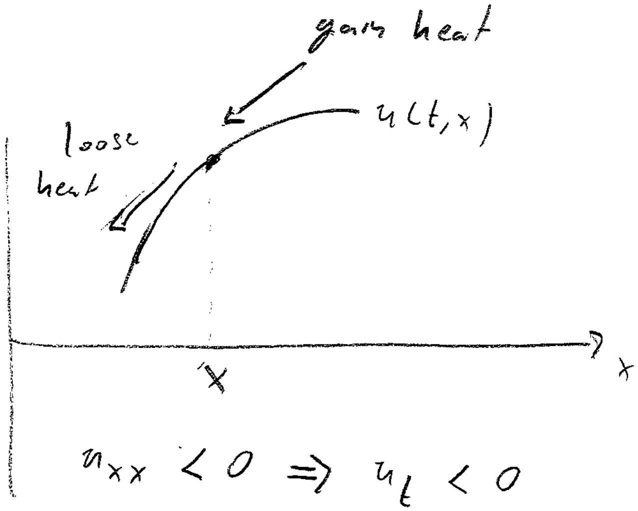
$$u = u(t, x).$$

The temperature dynamics is described by the <u>heat equation</u>

$$u_t - k u_{xx} = 0 \quad , \quad \text{notation } u_t = \frac{\partial u}{\partial t}, \; u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

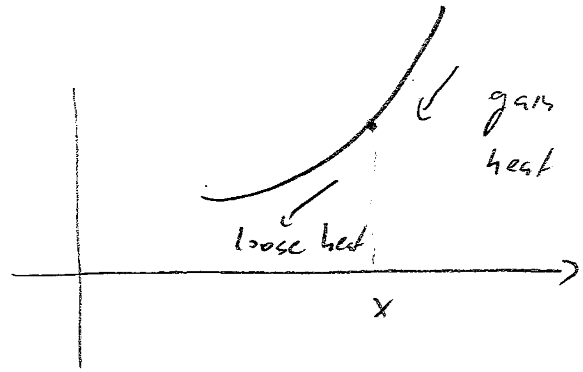
where $k > 0$ is a constant depending on the nature of the rod known as thermal diffusivity.

We can interpret this equation physically upon imagining the graph of u for a fixed time t



$u_{xx} < 0 \Rightarrow u_t < 0$

temperature at x will (instanteneously) decrease



$u_{xx} > 0 \Rightarrow u_t > 0$

temperature at x will (instanteneously) increase.

Suppose that u is a solution to the heat equation, and suppose that $t=0$ corresponds to "now". The temperature at later times, $u(t,x)$, will depend on the temperature at $t=0$. Thus we cannot completely calculate $u(t,x)$ without knowing $u(0,x)$. Mathematically, this means that the solution u contains <u>undetermined functions</u> that must be determined with the knowledge of $u(0,x)$. (Compare to <u>ODEs</u>). Hence, to completely find u we must also be given

undetermined <u>constants</u>

$$u(0,x) = u_0(x)$$

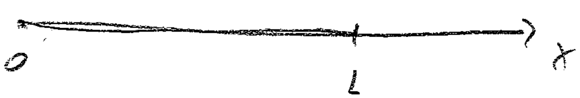
where u_0 is a known function of x only, called an initial condition (I.c)

Thus, the complete problem to find u is

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(0,x) = u_0(x) \end{cases}$$

A PDE + I.C. is know as an initial value problem

Suppose now that instead of an infinite rod we consider a rod of length L:



Assume that the temperature of the rod at the endpoints is kept constant by some external heat sources, so that $u(t,0) = T_1$, $u(t,L) = T_2$ where T_1 and T_2 are known numbers. These are boundary conditions, and in this case the problem of completely determining u becomes

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(0,x) = u_0(x) \\ u(t,0) = T_1, \; u(t,L) = T_2 \end{cases}$$

A PDE + I.C. + B.C. is known as an initial-boundary value problem.

Notice that we can choose different boundary conditions. For instance, suppose that the rod endpoints are insulated, so that no heat can be transferred across the endpoints. In this case the rate of change of the temperature across the endpoints has to be zero:

$$u_x(t,0) = 0 \quad \text{and} \quad u_x(t,L) = 0$$

and the initial-boundary value problem is
$$\begin{cases} u_t - k\,u_{xx} = 0 \\ u(0,x) = u_0(x) \\ u_x(t,0) = 0 = u_x(t,L) \end{cases}$$

Going back to the wave equation, we notice that to completely determine u we also need to be given the initial configuration of the string, which in this case is not only $u(0,x)$ but also $\partial_t u(0,x)$ (by Newton's law, to determine the motion of an object, we need to provide initial position and initial velocity).

The complete initial-boundary value problem for the wave equation is

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(0,x) = u_0(x), \quad \dfrac{\partial u}{\partial t}(0,x) = u_1(x) \\ u(t,0) = 0 = u(t,L) \end{cases}$$

where u_0 and u_1 are known functions (of x only)

Let's see another example

Suppose we want to study the electric and magnetic field in a region of space. Let E and B denote the electric and magnetic fields, respectively. These are vectors:

$$E = (E^1, E^2, E^3) \quad \text{and} \quad B = (B^1, B^2, B^3)$$

(If we used arrows for vectors $\vec{E} = E^1\vec{i} + E^2\vec{j} + E^3\vec{k}$)

(Notice that we do not employ any special notation for vectors, i.e., $\vec{E}$, $\underline{\underline{E}}$, etc)

E and B depend on time, which we denote by t, and space, which we denote by (x, y, z): So E and B are vector valued functions or vector functions or vector fields:

$$E = E(t, x, y, z) \quad \text{and} \quad B = B(t, x, y, z)$$

(i.e., $E^i = E^i(t, x, y, z)$, $B^i = B^i(t, x, y, z)$; Remark on notation)

Suppose that there exists a distribution of charge $\rho = \rho(t, x, y, z)$ and and electric current $J = (J^1, J^2, J^3)$, $J = J(t, x, y, z)$ in space

The behavior of E and B in the presence of ρ and J is described by Maxwell's equations,

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$ (Gauss law: the electric flux leaving a volume is proportional to the charge inside)

$$\nabla \cdot B = 0$$ (There are no magnetic monopoles; compare to Gauss law)

$$\frac{\partial B}{\partial t} = -\nabla \times E$$ (Faraday's law: voltage induced in a closed circuit is proportional to the rate of change of the magnetic flux it encloses — connection via Stoke's theorem)

$$\overbrace{\frac{\partial E}{\partial t}}^{\text{displacement current}} = \frac{1}{\mu_0 \varepsilon_0} \nabla \times B - \frac{1}{\varepsilon_0} J$$ (Ampère's law: the magnetic field induced around a closed loop is proportional to the electric current plus displacement current it encloses)

where: ε_0 = permittivity of free space $= 8.85 \cdot 10^{-12}$ F/m (farads per meter)

μ_0 = permeability of free space $= 4\pi \cdot 10^{-7}$ N/A² (newtons per Ampère square)

$\nabla \cdot$ = divergence = div, $\nabla \cdot (v^1, v^2, v^3) = \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z}$

$\nabla \times$ = curl or rotational = curl, $\nabla \times (v^1, v^2, v^3) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v^1 & v^2 & v^3 \end{bmatrix}$

$$= \left(\frac{\partial v^3}{\partial y} - \frac{\partial v^2}{\partial z}, \frac{\partial v^1}{\partial z} - \frac{\partial v^3}{\partial x}, \frac{\partial v^2}{\partial x} - \frac{\partial v^1}{\partial y} \right)$$

The first two equations can be written as

$$\frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial y} + \frac{\partial E^3}{\partial z} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \frac{\partial B^1}{\partial x} + \frac{\partial B^2}{\partial y} + \frac{\partial B^3}{\partial z} = 0$$

The last two equations are vector equations and each corresponds to three scalar equations

$$\frac{\partial B^1}{\partial t} = -\left(\frac{\partial E^3}{\partial y} - \frac{\partial E^2}{\partial z}\right), \quad \frac{\partial B^2}{\partial t} = -\left(\frac{\partial E^1}{\partial z} - \frac{\partial E^3}{\partial x}\right), \quad \frac{\partial B^3}{\partial t} = -\left(\frac{\partial E^2}{\partial x} - \frac{\partial E^1}{\partial y}\right)$$

(recall $\frac{\partial}{\partial t}(v^1, v^2, v^3) = \left(\frac{\partial v^1}{\partial t}, \frac{\partial v^2}{\partial t}, \frac{\partial v^3}{\partial t}\right)$; e.g. curves in multivariable calculus)

$$\frac{\partial E^1}{\partial t} = \frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial B^3}{\partial y} - \frac{\partial B^2}{\partial z}\right) - \frac{1}{\varepsilon_0} J^1, \quad \frac{\partial E^2}{\partial t} = \frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial B^1}{\partial z} - \frac{\partial B^3}{\partial x}\right) - \frac{1}{\varepsilon_0} J^2, \quad \frac{\partial E^3}{\partial t} = \frac{1}{\mu_0 \varepsilon_0}\left(\frac{\partial B^2}{\partial x} - \frac{\partial B^1}{\partial y}\right) - \frac{1}{\varepsilon_0} J^3$$

Suppose we know ρ and J. Then, to study E and B, we would like to solve the above equations, finding the vector fields E and B.

Each of Maxwell's equations are Partial Differential Equations (PDEs), and together they form a system of PDEs.