MATH 3120

Introduction to partial differential equations

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Abbreviations:

ODE = ordinary differential equation

PDE = partial differerential equation

HW = homework

LHS = left hand side

RHS = right hand side

w.r.t. = with respect to

$\Rightarrow$ = implies

Ex = example

Def = definition

Theo = theorem

Prop = proposition

$\square$ = end of a proof.

LHS := RHS means that the LHS is defined by the RHS

n d (e.g. 1d, 2d, ...) = n dimensional

iff = if and only if

What are partial differential equations and why to study them?

Recall that an ordinary differential equation (ODE) is an equation involving an unknown function of a single variable and some of its derivatives. For example,

$$\frac{dy}{dx} + y^2 = 0, \qquad \text{(unknown } y\text{, non-linear, 1st order)}$$

$$y'' + y' + y = 0, \qquad \text{(unknown } y\text{, linear, 2nd order)}$$

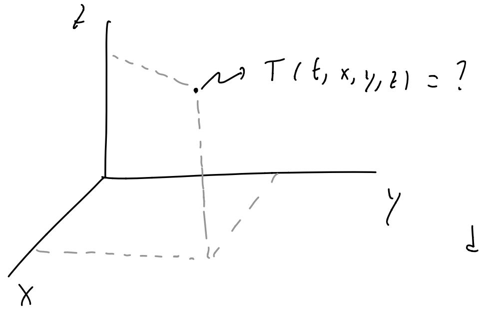
$$(x^2 - 1)\frac{d^2 u}{dx^2} + u = 0, \qquad \text{(unknown } u\text{, linear, 2nd order)}$$

are ODEs. We can also have systems of ODEs, i.e., a system of equations involving two or more unknown functions of a single variable and their derivatives. For example,

$$\begin{cases} \dfrac{dy}{dt} + x = 0 \\ \dfrac{dx}{dt} - y = 0 \end{cases} \qquad \text{(unknowns: } y \text{ and } x\text{, linear, 1st order)}$$

$$\begin{cases} u'' + v^2 = 0 \\ v'' + w - u' = 0 \\ w'' + w' + wv = 0 \end{cases} \qquad \text{(unknowns: } u, v, w\text{, non-linear, 2nd order)}$$

are systems of ODEs. As we learn in ODE courses, one typically studies ODEs because many phenomena in science and engineering are model with ODEs. A limitation of ODEs, however, is that they are restricted to functions of a single varible, whereas many important phenomena are described by functions of several variables. For instance, suppose we want to describe the temperature T in a room. It will in general be different at different positions in the room, so T is a function of (x, y, z). T can also change over time, thus $T = T(t, x, y, z)$. An equation involving T and its



derivatives can then have derivatives with respect to any of the variables t, x, y, or z, which will be partial derivatives, $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. This will be a partial differential equation. Formally:

Def. A partial differential equation (PDE) is an equation involving an unknow function of two ore more variables and some of its (partial) derivatives. A system of PDEs is a system of equations involving two or more unknown functions of two ore more variables and some of their (partial) derivatives. A solution to a PDE (or system) is a function that verifies the PDE.

Notation. Since most of the time we will be dealing with functions of several variables, the derivatives will be partial derivatives, but we will often omit the word "partial", referring simply to "derivatives." We will also often omit "system", and use PDE to refer to both a single equation and systems of PDEs.

Besides applications to science and engineering, PDEs are also used in many branches of mathematics, such as in complex analysis or geometry (see in particular Ricci flow and the Poincaré conjecture). PDEs are also studied in mathematics for their own sake, i.e., from a "pure" point of view.

Examples and notation

We will now give examples of PDEs. Along the way, we will introduce some notation that will be used throughout.

Remark. As it was the case for ODEs, when we introduce a PDE strictly speaking we have to specify where the equation is defined. We will often ignore this for the time being until we get to some more formal aspects of PDE theory.

Laplace's equation:

$$\Delta u = 0,$$

where Δ is the Laplacian operator defined by

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

so explicitly Laplace's equation reads:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

We will often denote coordinates in $\mathbb{R}^3$ by (x^1, x^2, x^3), in which case we write Δ as $\Delta = \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2}$.
We write expression of the form $u = u(x^1, x^2, x^3)$ to indicate the variables that a function depends on, e.g., in this case that

u is a function of x^1, x^2, and x^3. We can also consider Laplace's equation for a function of $x^1, x^2, \dots, x^n$, for some arbitrary n, $u = u(x^1, x^2, \dots, x^n)$, in which case

$$\Delta := \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \dots + \frac{\partial^2}{\partial (x^n)^2},$$

so Laplace's equation reads

$$\Delta u = \frac{\partial^2 u}{\partial (x^1)^2} + \frac{\partial^2 u}{\partial (x^2)^2} + \dots + \frac{\partial^2 u}{\partial (x^n)^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial (x^i)^2} = 0.$$

Laplace's equation has many applications. Typically, u represents the density of some quantity (e.g., a chemical concentration). Closely related to Laplace's equation is the Poisson equation:

$$\Delta u = f,$$

where f is a given function.

Heat equation or diffusion equation

$$\partial_t u - \Delta u = 0.$$

This equation has many applications. For example u can represent the temperature, so $u(t, x^1, x^2, x^3)$ is the temperature at the point (x^1, x^2, x^3) at instant t. More generally, u can represent the concentration of some quantity that diffuses over time.

Notation. Throughout these notes, we will use t to denote a time variable, unless otherwise specified.

Remark. The heat equation is also written as $\partial_t u - k\Delta u = 0$, where k is a constant known as diffusivity. In most of these notes, we will ignore physical constants in the equations, setting them equal to 1.

Wave equation

$$u_{tt} - \Delta u = 0$$

(Here we recall the notation $u_t = \partial_t u = \frac{\partial u}{\partial t}$, $u_{tt} = \partial^2_{tt} u = \frac{\partial^2 u}{\partial t^2}$ etc.). This equation describes a wave propagating in a medium (e.g., a radio wave propagating in space); u is the amplitude of the wave.

Sometimes one writes $u_{tt} - c^2 \Delta u = 0$ where the constant c is the speed of propagation of the wave (we will see later on why c is indeed the speed of propagation).

Schrödinger's equation

$$i\frac{\partial \Psi}{\partial t} + \Delta \Psi + V \Psi = 0,$$

where i is the complex unit $i^2 = -1$, $V = V(t, x^1, x^2, x^3)$ is a known function called the potential (whose specific form depends on the problem we are studying), and the unknown function Ψ, called the wave-function, is a complex function, i.e,

$$\Psi = u + iv,$$

where u and v are real valued functions.

The Schrödinger equation is the fundamental equation of quantum mechanics.

Burgers' equation

$$u_t + u u_x = 0.$$

Burgers' equation has applications in the study of shock waves.

Maxwell's equations

$$\begin{cases} \partial_t E - \operatorname{curl} B = -J, \\ \partial_t B + \operatorname{curl} E = 0, \\ \operatorname{div} E = \rho, \\ \operatorname{div} B = 0, \end{cases}$$

where the E and B are vector fields that are the unknown functions (or vector valued functions), so they have three components each:

$$E = (E^1, E^2, E^3),$$

$$B = (B^1, B^2, B^3),$$

div and curl are the divergence and curl operators, sometimes written as $\nabla\cdot$ and $\nabla\times$, respectively (curl is also called the rotational). Let us recall the definition of these operators: for any vector field $X = (X^1, X^2, X^3)$, we have

$$\operatorname{div} X := \partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3,$$

and

$$\operatorname{curl} X := \left(\partial_2 X^3 - \partial_3 X^2, -\partial_1 X^3 + \partial_3 X^1, \partial_2 X^3 - \partial_3 X^2 \right),$$

where we have introduced the following notation:

$$\partial_i := \frac{\partial}{\partial x^i}.$$

E and B represent the electric and magnetic fields, respectively. ρ represents the charge density and J the current density, which are given.

Maxwell's equations are the fundamental equations of electromagnetism.

<u>Notation</u>. Note that above we did not denote vectors with an "arrow," i.e., $\vec{E}$ and $\vec{B}$, as usually done in calculus. We will avoid using arrows for vectors – it will always be clear from the context if a quantity is a scalar, a vector field, etc. We also denote the components or entries of a vector with superscrips and not with subscripts as usually in calculus (i.e., X^i and not X_i, but see below for exceptions).

Similarly, we will denote points in space by a single letter without an arrow, e.g., $X = (x^1, x^2, x^3)$ in $\mathbb{R}^3$, or more generally $x = (x^1, x^2, x^3, \dots, x^n)$ in $\mathbb{R}^n$. So, sometimes we write expressions like $u = u(t, x)$ instead of $u = u(t, x^1, x^2, x^3)$.

Notation. The curl can be written in a compact for as

$$\underbrace{(\operatorname{curl} X)^i}_{\text{meaning the } i\text{th component of the vector curl } X} = \varepsilon^{ijk} \partial_j X_k .$$

In this expression, the following convention is adopted. ε is the totally anti-symmetric symbol, defined as

$$\varepsilon^{ijk} := \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases}$$

E.g., $\varepsilon^{123} = 1$, $\varepsilon^{231} = 1$, $\varepsilon^{213} = -1$, $\varepsilon^{112} = 0$. X_k means X^k, but we write it here with a subscript because of the following summation convention which will be used throughout:

> When an index (such as i, j, etc.) appears repeated in an expression, once upstairs and once downstairs, it is summed over its range.

E.g., we can write the divergence as

$$\operatorname{div} X = \partial_i X^i = \sum_{i=1}^{3} \partial_i X^i$$

$$= \partial_1 X^1 + \partial_2 X^3 + \partial_3 X^3 .$$

Remark. We will give another interpration to X_k (i.e., X^k but with the index downstairs) which will make our conventions more systematic, later on.

In the expression for curl, for example:

$$(\operatorname{curl} X)^2 = \varepsilon^{2jk} \partial_j X_k$$

$$= \varepsilon^{213} \partial_1 X_3 + \varepsilon^{231} \partial_3 X_1$$

$$= -\partial_1 X_3 + \partial_3 X_1 .$$

We also sometimes use the notation

$$\operatorname{curl}^i X = (\operatorname{curl} X)^i .$$

Euler and Navier-Stokes equations

$$\begin{cases} \partial_t \rho + (u \cdot \nabla)\rho + \rho \operatorname{div} u = 0 \\ \rho(\partial_t u + (u \cdot \nabla) u) + \nabla p = \mu \Delta u \end{cases}$$

These equations describe the motion of a fluid. The first equation is sometimes called the continuity equation (conservation of mass) and the second one the momentum equation (conservation of momentum).

$\rho = \rho(t,x)$ is a scalar function representing the fluid's density and $u = u(t,x)$ is a vector field representing the fluid's velocity. ρ and u are the unknowns. p is a given function of ρ, i.e., $p = p(\rho)$ (e.g., $p(\rho) = \rho^2$). p represents the pressure of the fluid. $\mu \geq 0$ is a constant known as the viscosity of the fluid. ∇ is the gradient operator; recall that $\nabla f := (\partial_1 f, \partial_2 f, \partial_3 f)$, where f is a scalar function, so the i^{th} component reads $(\nabla f)^i = \partial_i f$; we also write $\nabla^i f$ for $(\nabla f)^i$.

$u \cdot \nabla$ is the operator

$$u \cdot \nabla = u^i \partial_i = u^1 \partial_1 + u^2 \partial_2 + u^3 \partial_3 .$$

When $u \cdot \nabla$ acts on a vector field it does so componentwise. Δ also acts on a vector field componentwise.

These equations are known as the Navier-Stokes equations if $\mu > 0$ and Euler equations if $\mu = 0$. They are the fundamental equations of hydrodynamics.

In models where the density is assumed to be constant, in which case we take $\rho \equiv 1$, we have the incompressible Euler or Navier-Stokes equations:

$$\begin{cases} \operatorname{div} u = 0 \\ \partial_t u + (u \cdot \nabla) u + \nabla p = \mu \Delta u \end{cases}$$

In this case, however, it is no longer assumed that $p = p(\rho)$, and p is given by some other expression (we will see this later).

Other examples

There are many other important PDEs that we will not have time to discuss. We mention a few more of them, without writing them explicitly:

Einstein's equations: fundamental equations of general relativity.

Yang-Mills equations: fundamental equations of quantum field theory.

Black-Scholes equation: models the price of European options.

Remark. The concepts of the order of a PDE and of homogeneous vs. non-homogeneous PDEs are defined similarly to their analogues in ODEs. We will define linear and non-linear PDEs later on, but this definition is also similar to ODEs and readers should be able to identify which of the above examples are linear or non-linear PDEs.

Theory and examples. Before investigating more general and theoretical aspects of PDEs, it is useful to first consider a few specific equations that can be solved explicitly. Thus, at the beginning will be more computational and equation-specific. Later on we will consider more robust aspects of the general theory of PDEs.

The Scrödinger equation and the method of separation of variables

If we write the physical constants, the Schrödinger equation can be written as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta\Psi + V\Psi,$$

where $\hbar$ is Planck's constant, μ is a constant called the mass, and $i^2=-1$. $V=V(t,x): \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is a given function called the potential and $\Psi = \Psi(t,x): \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is the unknown function, called the wave function, and $\mathbb{C}$ is the set of complex numbers.

Notation. We have a function depending on time and space, i.e., t and x, we will often write its domain as $\mathbb{R} \times \mathbb{R}^3$ instead of $\mathbb{R}^4$, to emphasize that $t \in \mathbb{R}$ is the time variable and $x \in \mathbb{R}^3$ is the space variable.

The Schrödinger equation describes the evolution of a particle of mass μ interacting with a potential V, according to the laws of quantum mechanics.

Physical interpretation of Ψ. Given a subset $\mathcal{U} \subseteq \mathbb{R}^3$, the integral

$$\int_{\mathcal{U}} |\Psi(t,x)|^2 \, dx$$

is the probability of finding the particle in the region $\mathcal{U}$ at time t, where $|\Psi|^2$ is the square of the absolute value of Ψ:

$$|\Psi|^2 = \Psi^* \Psi,$$

where Ψ^* is the complex conjugate of Ψ.

Note that one must have

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2 \, dx = 1.$$

This latter condition can always be satisfied, upon multiplying Ψ by a suitable constant, as long as

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2 \, dx < \infty.$$

<u>Notation</u>. Above and throughout, we use dx to denote the volume element in $\mathbb{R}^n$, i.e.,

$$dx = dx^1 \, dx^2 \cdots dx^n,$$

so in particular in $\mathbb{R}^3$

$$dx = dx^1 \, dx^2 \, dx^3.$$

We denote the integral of a function f over a region $\mathcal{U} \subseteq \mathbb{R}^n$ by $\int_{\mathcal{U}} f(x)\, dx$ or sometimes simply $\int_{\mathcal{U}} f\, dx$, i.e., we don't write $\iint_{\mathcal{U}} \cdots \int f\, dx$ as in multivariable calculus.

Separation of variables for a time independent potential

We now suppose that V does not depend on t:

$$V = V(x).$$

One of the simplest methodos to try to solve a linear PDE is called the method of separation of variables. We will apply this method here. Further applications of the method will be given as HW.

The method of separation of variables consists in supposing that the unknown function is a product of functions of a single variable (this does not need to be always true, but it is a good starting point, and it will work here).

Thus, we suppose that

$$\Psi(t,x) = T(t)\,\psi(x).$$

Plugging this into the Schrödinger equation gives

$$\underbrace{i\hbar \frac{T'}{T}}_{\text{function of } t \text{ only}} = \underbrace{-\frac{\hbar^2}{2\mu}\frac{\Delta\psi}{\psi} + V}_{\text{function of } x \text{ only}}.$$

Since LHS = function of t only, RHS = function of x only, the only way to have LHS = RHS is if both sides equal a constant E:

$$i\hbar\frac{T'}{T} = E \Rightarrow i\hbar T' = ET$$

$$-\frac{\hbar^2}{2\mu}\frac{\Delta\psi}{\psi} + V = E \Rightarrow -\frac{\hbar^2}{2\mu}\Delta\psi + V\psi = E\psi.$$

The first equation has solution $T(t) = e^{-\frac{iE}{\hbar}t}$, where we ignored (and often will) an arbitrary constant of integration since the PDE is linear. The second equation is known as the time-independent Schrödinger equation.

The time independent Schrödinger equation for a radially symmetric potential.

We now focus on

$$-\frac{\hbar^2}{2\mu}\Delta\psi + V\psi = E\psi$$

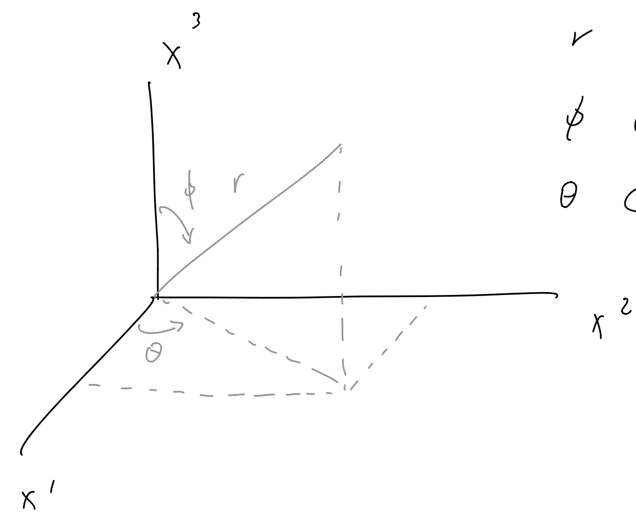
We make another assumption on V. We suppose that it is radially symmetric, i.e.,

$$V(x) = V\left(\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right),$$

or, in spherical coordinates, that

$$V(r,\phi,\theta) = V(r),$$

where (r, ϕ, θ) are spherical coordinates:



$r \in [0, \infty)$
$\phi \in [0, \pi]$
$\theta \in [0, 2\pi)$

We will work in spherical coordinates, so $\psi = \psi(r, \phi, \theta)$. The Laplacian in spherical coordinates reads

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2},$$

where

$$\Delta_{S^2} := \partial_\phi^2 + \frac{\cos\phi}{\sin\phi}\partial_\phi + \frac{1}{\sin^2\phi}\partial_\theta^2$$

is ~~the~~ called the Laplacian on the (unit) sphere.

We apply separation of variables again:

$$\psi(r, \phi, \theta) = R(r)\, Y(\phi, \theta).$$

Plugging in the equation and using Δ in spherical coordinates

$$\underbrace{-\frac{\hbar^2}{2\mu}\frac{r^2}{R}\left(R'' + \frac{2}{r}R\right) + (V - E)\,r^2}_{\text{function only of } r} = \underbrace{\frac{\hbar^2}{2\mu}\frac{\Delta_{S^2} Y}{Y}}_{\text{function only of } (\phi, \theta)}$$

$\Rightarrow$ LHS = RHS = constant = $-a$. Thus

$$-\frac{\hbar^2}{2\mu}\left(R'' + \frac{2}{r}R'\right) + \left(V + \frac{a}{r^2}\right)R = E R \quad \text{(radial eq.)}$$

$$\frac{\hbar^2}{2\mu}\Delta_{S^2} Y = -a\, Y \quad \text{(angular eq.)}$$

Remark. Note that we do not know at this point the values of the constants E and a.

The angular equation

Using the formula for Δ_{S^2}, the angular equation reads

$$\partial_\phi^2 Y + \frac{\cos\phi}{\sin\phi}\partial_\phi Y + \frac{1}{\sin^2\phi}\partial_\theta^2 Y = -\frac{2a\mu}{\hbar^2} Y.$$

Apply separation of variables again:

$$Y(\phi,\theta) = \Phi(\phi)\,\Theta(\theta),$$

so

$$\underbrace{-\frac{\Theta''}{\Theta}}_{\text{function of } \theta \text{ only}} = \underbrace{\sin^2\phi\,\frac{\Phi''}{\Phi} + \sin\phi\cos\phi\,\frac{\Phi'}{\Phi} + \frac{2a\mu\sin^2\phi}{\hbar^2}}_{\text{function of } \phi \text{ only}}$$

$\Rightarrow$ LHS = RHS = constant = b.

Then

$$\Theta'' = -b\,\Theta$$

$$\sin^2\phi\,\Phi'' + \sin\phi\cos\phi\,\Phi' + \frac{2ar\sin^2\phi}{\hbar^2}\,\Phi = b\,\Phi$$

Since the coordinates θ and $\theta + 2\pi$ represent the same point in $\mathbb{R}^3$, Θ must be periodic:

$$\Theta(\theta + 2\pi) = \Theta(\theta).$$

Solutions to the Θ equation depend on the sign of b. If $b < 0$ then the only periodic solution is the zero solution. Thus $b \geq 0$ and solutions are linear combinations of $\cos(\sqrt{b}\,\theta)$ and $\sin(\sqrt{b}\,\theta)$, and we must have $\sqrt{b}$ = integer for 2π-periodicity.

Thus we can write

$$b = m^2, \quad m \in \mathbb{Z},$$

which determines b, and we find

$$\boxed{\Theta(\theta) = e^{im\theta},}$$

$m \in \mathbb{Z}$.

We now investigate the Φ equation. Using the chain rule and $b = m^2$, it can be written as

$$\frac{\sin\phi}{\Phi}\frac{d}{d\phi}\left(\sin\phi\,\frac{d\Phi}{d\phi}\right) - m^2 = -\lambda\sin^2\phi,$$

where

$$\lambda := \frac{2\mu}{\hbar^2}a.$$

To solve the Φ equation, we make a change of variables:

$$x := \cos\phi, \quad 0 \leq \phi \leq \pi.$$

(not to be confused with a point $x \in \mathbb{R}^3$).

Using the chain rule, the equation becomes:

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0,$$

which is know as Legendre's equation. To solve it, we seek a solution of the form

$$\Phi(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P(x)}{dx^{|m|}},$$

where P is a solution to

$$(1-x^2)\frac{d^2 P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0.$$

It is an exercise to verify that if P solves the above equation, then Φ, as given above in terms of P, solves the Legendre equation. So it suffices to find P.

We seek a power series solution:

$$P(x) = \sum_{k=0}^{\infty} a_k x^k .$$

Plugging in:

$$\underbrace{(1-x^2)\sum_{k=0}^{\infty} k(k-1)\, a_k x^{k-2} - 2x \sum_{k=0}^{\infty} k\, a_k x^{k-1} + \lambda \sum_{k=0}^{\infty} a_k x^k}_{} = 0$$

$$= \sum_{k=0}^{\infty} \left[(k+2)(k+1)\, a_{k+2} - \left(k(k+1) - \lambda \right) a_k \right] x^k$$

This implies the recurrence relation:

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+1)(k+2)} a_k, \quad k = 0, 1, 2 \ldots$$

a_0, a_1 arbitrary.

What about convergence? Writing the sum as separate linearly independent even and odd powers:

$$\lim_{k \to \infty} \left| \frac{a_{k+2} \, x^{k+2}}{a_k \, x^k} \right| = |x|^2,$$

so the series converges for $|x| < 1$. Testing the endpoints $x = \pm 1$ (i.e., $\phi = 0$ and $\phi = \pi$):

$$P(\pm 1) = \pm \sum_{k=0}^{\infty} a_k .$$

From the recurrence relation

$$a_{k+2} = \frac{k^2 + O(k)}{k^2 + O(k)} a_k$$

$$= \frac{k^2 + O(k)}{k^2 + O(k)} \frac{(k-2)^2 + O(k-2)}{(k-2)^2 + O(k-2)} a_{k-2}$$

$$= \dots$$

$$= \begin{cases} \dfrac{k^{k+2} + O(k^{k+1})}{k^{k+2} + O(k^{k+1})} a_0, & k \text{ even}, \\[2ex] \dfrac{k^{k+1} + O(k)}{k^{k+1} + O(k)} a_1, & k \text{ odd}. \end{cases}$$

Therefore, $\lim_{k \to \infty} a_k \neq 0$ and $P(\pm 1)$ diverges

unless $a_k = 0$ for $k > l$ for some l, i.e.,

$$a_{l+2} = \frac{l(l+1) - \lambda}{(l+1)(l+2)} a_l = 0,$$

with $a_l \neq 0$. Then

$$\lambda = l(l+1), \quad l = 0, 1, 2, \dots,$$

which determines λ and thus the constant a.

We see that we obtained a family $\{P_\ell\}$ of solutions parametrized by ℓ. Note that P_ℓ is a polynomial of degree ℓ, thus $\Phi = 0$ for $|m| > \ell \Rightarrow |m| \leq \ell$. We write $m = m_\ell$ to stress that the allowable values of m depend on ℓ. The P_ℓ's are called Legendre polynomials. We then obtain a family $\{\Phi_{\ell, m_\ell}\}$ of solutions. For example

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 1 - 3x^2,$$

$$\Phi_{00}(x) = 1, \quad \Phi_{10}(x) = x, \quad \Phi_{1,\pm 1}(x) = (1 - x^2)^{1/2},$$

where we chose a_0 and a_1 conveniently to obtain integer coefficients.

We have to go back to the variable ϕ. Denote:

$$F_{\ell,m_\ell}(x) := \frac{d^{|m|}P(x)}{dx^{|m|}}.$$

Then, recalling $x = \cos\phi$

$$\boxed{\Phi_{\ell,m_\ell}(\phi) = \sin^{|m_\ell|}\phi \; F_{\ell,m_\ell}(\cos\phi),}$$

$\ell = 0, 1, 2, \dots$, $|m_\ell| \leq \ell$. The functions F_{ℓ,m_ℓ} are called associated Legendre functions.

We finally obtain the following family of solutions to the angular equation:

$$\boxed{Y_{\ell,m_\ell}(\phi,\theta) = e^{im_\ell\theta}\sin^{|m_\ell|}\phi \; F_{\ell,m_\ell}(\cos\phi),}$$

$\ell = 0, 1, 2, \dots$, $|m_\ell| \leq \ell$. The functions Y_{ℓ,m_ℓ} are called spherical harmonics.

Note that now that we found the constant a, the Y equation reads

$$\Delta_{S^2} Y_{\ell, m_\ell} = -\ell(\ell+1) Y_{\ell, m_\ell}$$

which is an eigenvalue problem for the Laplacian on the sphere, whose solution is given by the spherical harmonics.

<u>Remark</u>. Spherical harmonics and Legendre polynomials have many applications in physics.

<u>The radial equation</u>

The radial equation can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}(E-V)R = \ell(\ell+1)\frac{R}{r^2}.$$

Everything we did so far holds for a general $V(r)$. But in order to solve the radial equation, we need to specify $V(r)$. We henceforth assume that V is the potential describing

The electromagnetic interaction of an electron and a nucleus:

$$V(r) = -\frac{Z e^2}{4\pi \varepsilon_0 r},$$

Z = nuclear charge, $-e$ = electron charge, ε_0 = vacuum permittivity.

Let us begin showing that the constant E is real. Multiplying the equation by $r^2 R^*$ and integrating from 0 to ∞:

$$\underbrace{\int_0^\infty R^* \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) dr}_{\text{integrate by parts}} - \frac{2\mu}{\hbar^2}\int_0^\infty V\, |R|^2 r^2\, dr - \ell(\ell+1)\int_0^\infty |R^2|\, dr$$

$$= -\frac{2\mu}{\hbar^2} E \int_0^\infty |R|^2 r^2\, dr$$

$$= -\int_0^\infty \frac{dR^*}{dr}\frac{dR}{dr} r^2\, dr + \underbrace{R^* \frac{dR}{dr} r^2 \Big|_0^\infty}_{=0}$$

$= 0$ for R, R' decaying sufficiently fast for large r

$$= -\int_0^\infty \left(\left(\frac{dR_R}{dr}\right)^2 + \left(\frac{dR_C}{dr}\right)^2\right) dr \quad \text{for } R = R_R + i R_C$$

Thus, we conclude that E is real. Let us next show that $E < 0$. For $r \gg 1$,

$$\frac{d^2R}{dr^2} \approx -\frac{2\mu}{\hbar^2} E R \Rightarrow \underbrace{r\frac{d^2R}{dr^2}}_{\approx\, r\frac{d^2R}{dr^2} + 2\frac{dR}{dr} = \frac{d^2}{dr^2}(rR)} \approx -\frac{2\mu}{\hbar^2} E(rR)$$

$$\Rightarrow \frac{d^2}{dr^2}(rR) \approx -\frac{2\mu E}{\hbar^2}(rR),$$ which has (approximate)

solution $rR \approx e^{\pm\sqrt{-\frac{2\mu}{\hbar^2}E}\, r}$. Thus, if $E > 0$, then R is a complex function satisfying $|rR| \approx 1$ and

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2\, dx = \int_{\mathbb{R}^3} |Y(\phi,\theta)|^2 \, |R(r)|^2 \, r^2 \sin\phi \, d\phi \, d\theta \, dr$$

$$= \int_0^{\pi}\int_0^{2\pi} |Y(\phi,\theta)|^2 \sin\phi \, d\phi \, d\theta \int_0^{\infty} |R(r)|^2 r^2 \, dr = \infty$$

since $r^2|R|^2 \approx 1$ for large r. Thus, $E < 0$.

Since $E < 0$, we can define the following real numbers:

$$\beta^2 = -\frac{2\mu E}{\hbar^2}, \quad \gamma = \frac{\mu Z e^2}{4\pi \varepsilon_0 \hbar^2 \beta}.$$

We make the change of variables $\rho = 2\beta r$, so that the equation for $R = R(\rho)$ becomes:

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right) = \left(-\frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2} + \frac{\gamma}{\rho}\right) R.$$

We will solve this equation using power series. However, it is an exercise to show that a direct application of the method, i.e., $R(\rho) = \sum_{k=0}^{\infty} a_k \rho^k$, does not work. To get a better idea of how to find solutions, we first consider $\rho \gg 1$, so

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{dR}{d\rho}\right) \approx -\frac{R}{4}.$$

Looking for $R(\rho) \approx e^{A\rho}$ and plugging in, we find $A = -1/2$, $R(\rho) \approx e^{-\frac{1}{2}\rho}$. This suggests looking for solutions of the form

$$R(\rho) = e^{-\frac{\rho}{2}} G(\rho).$$

Plugging in, we find that G satisfies

$$\frac{d^2G}{d\rho^2} + \left(\frac{2}{\rho} - 1\right)\frac{dG}{d\rho} + \left(\frac{r-1}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right)G = 0.$$

We seek a power series solution of the form

$$G(\rho) = \rho^s \sum_{k=0}^{\infty} a_k \rho^k = \sum_{k=0}^{\infty} a_k \rho^{k+s},$$

where s is to be determined. Plugging in gives:

$$0 = \left(s(s+1) - \ell(\ell+1)\right) a_0 \rho^{s-2} +$$

$$\sum_{k=0}^{\infty}\left[\left((s+k+1)(s+k+2) - \ell(\ell+1)\right)a_{k+1} - (s+k+1-r)a_k\right]\rho^{s+k-1}.$$

Vanishing of the first term gives $s(s+1) - \ell(\ell+1) = 0$

$\Rightarrow$ $s = \ell$ or $\underbrace{s = -(\ell+1)}$.

discarded as otherwhise $G(0)$ is not defined.

Using $s = \ell$ we then find

$$a_{k+1} = \frac{k+\ell+1-r}{(k+\ell+1)(k+\ell+2) - \ell(\ell+1)} a_k.$$

Using the ratio test we can see that the series converges for any ρ. However, the above recurrence relation also gives

$$a_{k+1} = \frac{k + \cdots}{k^2 + \cdots} a_k = \frac{1 + \cdots}{k + \cdots} a_k = \frac{1 + \cdots}{k + \cdots} \frac{1 + \cdots}{(k-1) + \cdots} a_{k-1}$$

$$= \frac{1 + \cdots}{k(k-1) \cdots (k-j) + \cdots} a_{k-j},$$

and we conclude that $G(\rho)$ is asymptotic to $\rho^l e^\rho$. This implies $R(\rho) = e^{-\frac{\rho}{2}} G(\rho) \approx \rho^l e^{\frac{\rho}{2}}$ which then gives $\int_{\mathbb{R}^3} |\Psi(t,x)|^2 dx = \infty$, unless the series for G terminates, i.e, for some k, $k + l + 1 - \gamma = 0$ $\Rightarrow \gamma = k + l + 1$. In particular, γ has to be an integer: $\gamma = n$, $n = l+1, l+2, \ldots$ or $\gamma = n$, $n = 1, 2, 3, \ldots$, $l = 0, 1, 2, \ldots, n-1$. From the definitions of γ and β, we have found the values of the constant E:

$$E = E_n = - \frac{\mu Z e^4}{2(4\pi\varepsilon_0)^2 \hbar^2 n^2}, \quad n = 1, 2, 3, \ldots$$

We can then write $R = R_{n,\ell}$ as

$$\boxed{R_{n,\ell}(r) = e^{-\frac{Zr}{n\alpha_0}} \left(\frac{Zr}{n\alpha_0}\right)^{\ell} G_{n,\ell}\left(\frac{Zr}{n\alpha_0}\right), \quad \begin{array}{l} n = 1, 2, \dots \\ \ell = 0, \dots, n-1. \end{array}}$$

where $\alpha_0 = 4\pi\varepsilon_0 \hbar^2 / \mu e^2$. Our solutions ψ are then given by $\psi = \psi_{n,\ell,m_\ell}$: $\psi_{n,\ell,m_\ell} = R_{n,\ell} \, Y_{\ell,m_\ell}$, and

$$\boxed{\Psi_{n,\ell,m_\ell}(t,x) = A_{n,\ell,m_\ell} \, e^{-i\frac{E_n}{\hbar}t} \, \psi_{n,\ell,m_\ell}(x),}$$

where
$$n = 1, 2, 3, \dots$$
$$\ell = 0, 1, \dots, n-1,$$
$$m_\ell = -\ell, -\ell+1, \dots, 0, \dots, \ell-1, \ell,$$

and A_{n,ℓ,m_ℓ} are constants chosen such that

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2 \, dx = 1.$$

The numbers n, ℓ, m_ℓ are called quantum numbers. E_n can be shown to correspond to energy levels of the electron.

Remark. Because the Schrödinger equation is linear, any linear combination of solutions Ψ_{n,ℓ,m_ℓ} (for possibly different values of n, l, m) is also a solution.

Remark. Because the Schrödinger equation is an evolution equation (i.e., it involves $\frac{\partial}{\partial t}$), we might expect to be given initial conditions, as in ODEs. What we found above is a family of general solutions (like in ODEs), but given $\Psi(0, x)$ (i.e., $\Psi(t, x)$ at $t=0$) we can find a unique solution with the corresponding initial condition at $t=0$. We will talk more about initial conditions and initial value problems later on.

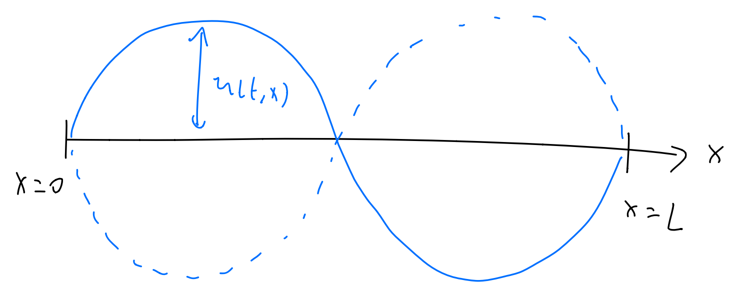
Separation of variables for the one-dimensional wave equation

Consider the wave equation in one dimension:

$$u_{tt} - c^2 u_{xx} = 0. \qquad (c \neq 0)$$

Notation. Whenever a PDE involves the time variable, by the dimension we always mean the spatial dimension. E.g., the one-dimensional wave equation (abbreviated 1d wave equation) is the wave equation for $u = u(t, x)$ with $x \in \mathbb{R}$.

We are interested in the case when the spatial variable belongs to a compact interval, e.g., $0 \leq x \leq L$, for some $L > 0$, and u vanishes at the extremities of the interval, i.e., $u(t, 0) = 0 = u(t, L)$. This the situation describing a string that can vibrate in the vertical direction with its ends fixed, with $u(t,x)$ representing the string amplitude at x at time t:



The conditions $u(t,0)=0$ and $u(t,L)=0$ are called boundary conditions because they are conditions imposed on the solution on the boundary of the domain where it is defined. Thus, the problem can be stated as

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } (0,\infty)\times(0,L) \\ & (\text{i.e. for } t\in(0,\infty),\ x\in(0,L)) \\ u(t,0) = 0 & \\ u(t,L) = 0 & \end{cases}$$

This is called a boundary value problem because it consists of a PDE plus boundary conditions. Sometimes we refer to a boundary value problem simply as PDE.

In the HW, you will be asked to show that applying seperation of variables we obtain the following family of solutions:

$$u_n(t,x) = \left(a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

where $n = 1, 2, 3, \ldots$ and a_n and b_n are arbitrary constants. Since the equation is linear, sums of the above functions are solutions, i.e.,

$$\sum_{n=1}^{N} u_n(t,x) = \sum_{n=1}^{N}\left(a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

is also a solution.

Because this holds for any N, we should be able to sum all way to infinity and still get a solution. In other words, the most general solution to the above boundary value problem is

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right),}$$

provided that this expression makes sense, i.e., the series converges.

Terminology. It often happens in PDEs that we have situations as the above, i.e., we have a formula for a would-be solution, but we do not know if the formula is in fact well-defined (e.g., we have a series that might not converge, or a function that might not be differentiable, etc.). "Solutions" of this type are called formal solutions. In other words, a formal solution is a candidate for a solution, but extra work must be done or further assumptions made in order to show that they are are in fact solutions.

The convergence of the above series cannot be decided without further information about the problem. This is because, as stated, the coefficients a_n and b_n in the formal solution are arbitrary, and it is not difficult to see that we can make different choices of these coefficients in order to make the series converge or diverge.

Therefore, we consider the above boundary value problem supplemented by <u>initial conditions</u>, i.e., we assumed given functions g and h defined on $[0, L]$ and look for a solution u such that

$$u(0,x) = g(x), \quad \partial_t u(0,x) = h(x), \quad 0 \leq x \leq L.$$

Similarly to what happens in ODEs, we expect that once initial conditions are given, we will no longer obtain a general solution but rather the unique solution that satisfies the initial conditions.

Remark. Note that any multiple of of the (formal) solution u will also be a (formal) solution. This is enconded in the arbitrariness of a_n and b_n, since if we multiple u by a constant A, we can simply redefine new coefficients as $\tilde{a}_n = A a_n$, $\tilde{b}_n = A b_n$. This freedom, however, is not present once we consider initial conditions, since if $u(0,x) = g(x)$, $\partial_t u(0,x) = h(x)$ then $Au(0,x) \neq g(x)$, $A\partial_t u(0,x) \neq h(x)$ (unless $A = 1$).

The previous remark suggests that the coefficients a_n and b_n should be determined from the initial conditions. Before investigating this, let us state the full problem. we want to find u such that

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } (0,\infty) \times [0,L], \\ u(t,0) = u(t,L) = 0, & t \geq 0, \\ u(0,x) = g(x), & 0 \leq x \leq L, \\ \partial_t u(0,x) = h(x), & 0 \leq x \leq L. \end{cases}$$

The above problem is called an <u>initial-boundary value problem</u> since it is a PDE with boundary conditions and initial conditions provided, although we sometimes call it simply a PDE.

Two initial conditions are prescribed, i.e., $u(0,x)$ and $\partial_t u(0,x)$, because the wave equation is second order in time. Note that g and h have to satisfy the following <u>compatibility conditions</u>:

$$g(0) = g(L) = h(0) = h(L) = 0.$$

We have already derived a formal solution to the wave equation satisfying the boundary conditions. It remains to investigate the initial conditions. Plugging $t=0$:

$$u(0,x) = g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$$

Differentiating u w.r.t. t and plugging $t=0$:

$$\partial_t u(0,x) = h(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi}{L} x\right).$$

Since g and h are in principle arbitrary, the above is essentially asking whether is possible to write an arbitrary function on $[0,L]$ as a series of sine functions with suitable coefficients. Or, rephrasing the question in a more appropriate form, we are asking: what are the functions on $[0,L]$ that can be written as a convergent series of sine functions with suitable coefficients? The functions for which this is true will provide us with a class of functions for which the above initial-boundary problem admits a solution.

The subject that investigates questions of this type is known as Fourier series. We will now make a digression to study Fourier series. After that, we will return to the wave equation.

Fourier series

We begin with the definition of Fourier series:

<u>Def</u>. Let $I = (-L, L)$ or $[-L, L]$, $L > 0$, and $f : I \to \mathbb{R}$ be integrable on I. The <u>Fourier series</u> of f, denoted $F.S.\{f\}$, is the series

$$F.S.\{f\}(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots$$

The coefficients a_n and b_n are called <u>Fourier coefficients</u>.

Remarks.

- $F.S.\{f\}$ is a series constructed out of f. We are not claiming that $F.S.\{f\} = f$. In fact, at this point we are not even claiming that $F.S.\{f\}$ converges (although we want to find conditions for which it converges, and for which $F.S.\{f\} = f$).

- The Fourier coefficients are well defined in view of the integrability of f.

- We introduced Fourier series for functions defined on an interval $[-L, L]$. This set-up is slightly different than what we encountered above for the wave equation, where we worked on the interval $[0, L]$. We will relate Fourier series on $[-L, L]$ with functions defined on $[0, L]$ later on.

- The Fourier series is a series of sine and cosine. The situation discussed above in the wave equation is a particular case where only sine is present (i.e., $a_n = 0$).

<u>Ex</u>: Find the Fourier series of

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi . \end{cases}$$

We compute:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = 0 \quad (\text{even-odd function})$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} \overbrace{f(x)}^{=1} \sin(nx)\, dx$$

$$= \frac{2}{\pi} \left(- \frac{\cos(nx)}{n} \right) \Bigg|_0^{\pi} = \frac{2}{\pi} \left(\frac{1}{n} - \frac{(-1)^n}{n} \right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} . \end{cases}$$

Thus:

$$F.S.\{f\}(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n} \right) \sin(nx)$$

$$= \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \dots \right).$$

Note that $f(0) = 1$ but $F.S.\{f\}(0) = 0$, so $F.S.\{f\} \neq f$.

Ex: Find the Fourier series of $f(x) = |x|$, $-1 \leq x \leq 1$.

Compute:

$$a_0 = \int_{-1}^{1} f(x)\, dx = 2\int_0^1 x\, dx = 1,$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx = 2\int_0^1 x \cos(n\pi x)\, dx = \frac{2}{\pi^2 n^2} ((-1)^n - 1),$$

$n = 1, 2, \dots$

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx = 0 \quad (\text{even} \cdot \text{odd})$$

Thus

$$F.S.\{f\}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left((-1)^n - 1 \right) \cos(n\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \left(\cos(\pi x) + \frac{1}{9} \cos(3\pi x) + \frac{1}{25} \cos(5\pi x) + \dots \right).$$

Piecewise functions

We begin with some definitions.

Def. Let $I \subseteq \mathbb{R}$ be an interval. A function $f: I \to \mathbb{R}$ is called k-times continuously differentiable if all its derivatives up to order k exist and are continuous. We denote by $C^k(I)$ the space of all k-times continuously differentiable functions on I. Note that $C^0(I)$ is the space of continuous functions on I. We denote by $C^\infty(I)$ the space of infinitely many times differentiable functions on I. Sometimes we say simply that "f is C^k" to mean that $f \in C^k(I)$. We write simply C^k for $C^k(I)$ if I is implicitly understood. C^∞ functions are also called smooth functions.

Ex: $e^x \in C^\infty(\mathbb{R})$, $|x| \in C^0(\mathbb{R})$. The function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is C^0, it is differentiable, but it is not C^1: this is because $f'(x)$ exists for every x (including $x=0$) but f' is not continuous at $x=0$.

Remark. Note that $C^k(I) \subseteq C^\ell(I)$ if $k \geq \ell$ and $C^\infty(I) = \bigcap_{k=0}^{\infty} C^k(I)$.

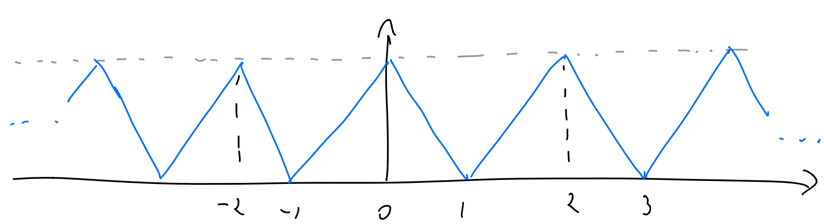
Def. Let $I \subseteq \mathbb{R}$ be an interval. We say that $f: I \to \mathbb{R}$ is a piecewise C^k function if f is C^k except possibly at a countable number of isolated points.

Ex: The functions $|x|$ and

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

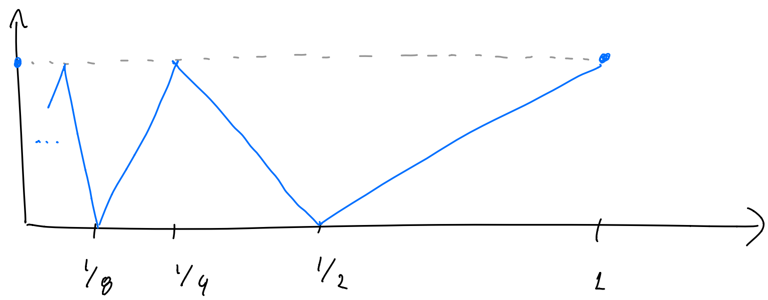
are piecewise smooth (C^∞) functions.

Ex: The function



is piecewise C^∞.

Ex: The function $f : [0,1] \to \mathbb{R}$ given by



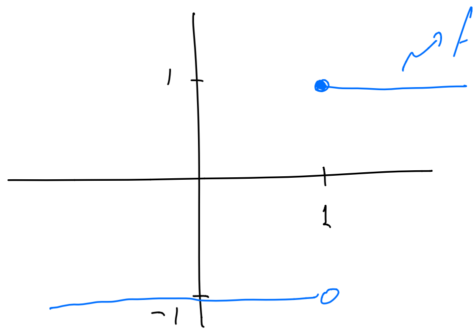
is not piecewise C^k because the set of points where it fails to be C^k are not isolated.

Convergence of Fourier series

Notation. We denote by $f(x^+)$ and $f(x^-)$ the right and left values of f at x, defined by $f(x^+) = \lim_{h \to 0^+} f(x+h)$, $f(x^-) = \lim_{h \to 0^-} f(x+h)$.

If f is continuous at x, then $f(x^+) = f(x^-) = f(x)$, but otherwise these values might differ.

Ex: For the function depicted below, $f(1^+) = 1$ and $f(1^-) = -1$:



$f(1^+) = 1,$

$f(1^-) = -1.$

Theo. Let f be a piecewise C^1 function on $[-L, L]$. Then, for any $x \in (-L, L)$:

$$F.S.\{f\}(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right),$$

and

$$F.S.\{f\}(\pm L) = \frac{1}{2}\left(f(-L^+) + f(L^-)\right).$$

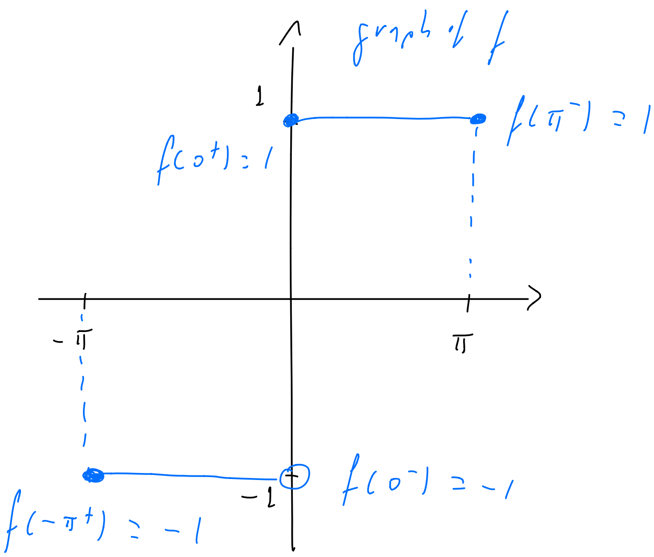
In particular, F.S. $\{f\}$ converges.

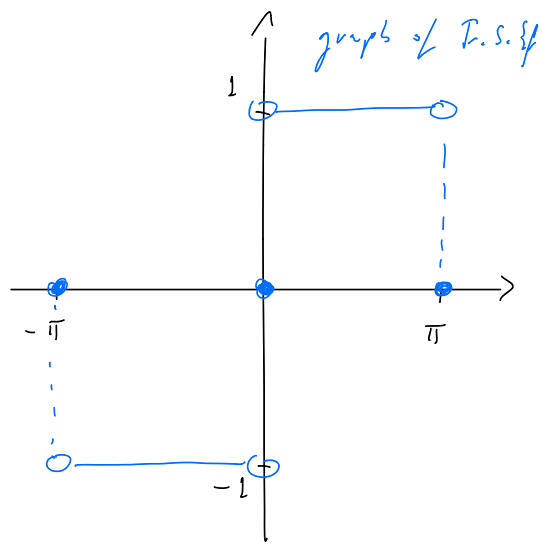
From the above theorem, we see that F.S. $\{f\}(x) = f(x)$ when f is continuous at x. Thus, if f is piecewise C^1 and C^0, we have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Ex: the graph $f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases}$ and

F.S.$\{f\}(x)$ below (note that f is piecewise C^1)





Ex: Since $|x|$ is continuous and piecewise C^1:

$$|x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x).$$

Next, we consider differentiation and integration of Fourier series term by term.

Theo. Let f be a piecewise C^2 and continuous function on $[-L,L]$, and assume that $f(-L)=f(L)$. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. More precisely, writing

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

we have

$$F.S.\{f'\}(x) = \sum_{n=1}^{\infty} \left(a_n \left(\cos\left(\frac{n\pi x}{L}\right)\right)' + b_n \left(\sin\left(\frac{n\pi x}{L}\right)\right)' \right)$$

$$= \sum_{n=1}^{\infty} \left(-\frac{a_n n\pi}{L} \sin\left(\frac{n\pi x}{L}\right) + \frac{b_n n\pi x}{L} \cos\left(\frac{n\pi x}{L}\right) \right).$$

In particular, if f' is continuous at x, we have

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} \left(-a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right) \right).$$

EX: To see that we cannot always differentiate a Fourier series term by term, consider $f(x)=x$, $-\pi \leq x \leq \pi$. Its Fourier series is

$$F.S.\{f\}(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx),$$

which converges for any x, but the term-by-term differentiated series, which is

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos(nx)$$

diverges for every x.

Theo. Let f be piecewise continuous on $[-L, L]$ with Fourier series

$$F.S.\{f\}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Then, for any $x \in [-L, L]$:

$$\int_{-L}^{x} f(t)\,dt = \int_{-L}^{x} \frac{1}{2}a_0\,dt + \sum_{n=1}^{\infty}\int_{-L}^{x}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)dt$$

Some intuition behind Fourier series

Let us make some comments about the way the Fourier series is defined. Given f defined on $[-L, L]$, our goal is to write:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Let us make an analogy with the following problem: given a vector $v \in \mathbb{R}^n$, we want to write

$$v = \sum_{i=1}^{n} c_i e_i,$$

where $\{e_i\}_{i=1}^{n}$ is an orthogonal basis of $\mathbb{R}^n$ (e.g., $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ in $\mathbb{R}^3$). In other words, we have to find the coefficients a_i. Since the vectors e_i are orthogonal

$$e_i \cdot e_j = 0 \quad \text{if } i \neq j,$$

where $\cdot$ is the dot product, a.k.a. inner product of vectors. Thus, for each $j = 1, \dots, n$:

$$e_j \cdot v = \sum_{i=1}^{n} c_i \underbrace{e_j \cdot e_i}_{\text{not zero only if } i=j} = c_j\, e_j \cdot e_j \Rightarrow c_j = \frac{v \cdot e_j}{e_j \cdot e_j}.$$

We want to do something similar to find the Fourier coefficients a_n and b_n. Consider the functions

$$E_0(x) = \frac{1}{2},$$

$$E_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \tilde{E}_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$$

Then: $$f = a_0 E_0 + \sum_{n=1}^{\infty} \left(a_n E_n + b_n \tilde{E}_n\right). \quad (*)$$

This is very similar to the case in $\mathbb{R}^n$. In fact, the space of piecewise C^k is a vector space, so (*) is an equality between vectors, although C^k is an infinite dimensional vector space so we need a basis with infinitely many vectors.

To find the Fourier coefficients the same way we found the coefficients c_j above, we need the analogue of the dot product for functions. It cannot be the usual product of functions, since the product of two functions is another function, whereas the dot product of two vectors is not another vector but a number. We also want our "dot product" for functions to have all the standard properties of the dot product of vectors. The relevant product for functions is defined below:

Def. Let $I \subseteq \mathbb{R}$ be an interval. The L^2 inner product, or simply inner product, of two functions $f, g : I \to \mathbb{R}$ is defined as

$$\langle f, g \rangle_{L^2} := \int_I f(x)\, g(x)\, dx$$

whenever the integral on the RHS is well-defined. We often write $\langle , \rangle$ for $\langle , \rangle_{L^2}$. The L^2 norm, or simply norm, of $f : I \to \mathbb{R}$ is defined as

$$\| f \|_{L^2} := \sqrt{\langle f, f \rangle}.$$

We sometimes write $\| \ \|$ for $\| \ \|_{L^2}$. We also write $\langle , \rangle_{L^2(I)}$ and $\| \ \|_{L^2(I)}$ if we want to emphasize the interval I.

It is a simple exercise to show that $\langle , \rangle_{L^2}$ has all the following properties, which are similar to the properties of the dot product:

1) $\langle f, g \rangle \in \mathbb{R}$ (when defined)

2) $\langle f, g \rangle = \langle g, f \rangle$

3) $\langle f, ag + bh \rangle = a\langle f, g \rangle + b\langle f, h \rangle$, $a, b \in \mathbb{R}$, f, g, h functions

4) $\langle f, 0 \rangle = 0$.

5) $\langle f, f \rangle \geq 0$. In particular, $\| \ \|_{L^2}$ is a real number if $\langle f, f \rangle < \infty$.

Remark. The dot product has the property $v \cdot v = 0 \Rightarrow v = 0$. This is not true for $\langle , \rangle_{L^2}$, as the example $f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$ shows. However, if f is continuous, then it is true that $\langle f, f \rangle_{L^2} = 0 \Rightarrow f = 0$.

Consider now $I = [-L, L]$ and let us go back to (*). A simple computation shows that

$$\langle E_n, E_m \rangle = 0 \text{ if } n \neq m, \quad \langle \tilde{E}_n, \tilde{E}_m \rangle = 0 \text{ if } n \neq m$$

$$\langle E_n, \tilde{E}_m \rangle = 0, \quad \langle \tilde{E}_n, \tilde{E}_n \rangle = L, \quad \langle E_n, E_n \rangle = \begin{cases} \frac{L}{2}, & n = 0 \\ L, & n > 0 \end{cases}$$

Taking the inner product of (*) with $E_m, \tilde{E}_m$, $m \geq 1$, and E_0, gives:

$$\langle f, E_m \rangle = a_0 \langle E_0, E_m \rangle + \sum_{n=1}^{\infty} \left(a_n \langle E_n, E_m \rangle + b_n \langle \tilde{E}_n, E_m \rangle \right)$$

$$= a_m \langle E_m, E_m \rangle = a_m L \Rightarrow a_m = \frac{\langle f, E_m \rangle}{L}$$

$$\langle f, E_0 \rangle = a_0 \langle E_0, E_0 \rangle + \sum_{n=1}^{\infty} \left(a_n \langle E_n, E_0 \rangle + b_n \langle \tilde{E}_n, E_0 \rangle \right)$$

$$= a_0 \langle E_0, E_0 \rangle = a_0 \frac{L}{2} \Rightarrow a_0 = \frac{2}{L} \langle f, E_0 \rangle$$

$$\langle f, \tilde{E}_m \rangle = a_0 \langle E_0, \tilde{E}_m \rangle + \sum_{n=1}^{\infty} \left(a_n \langle E_n, \tilde{E}_m \rangle + b_n \langle \tilde{E}_n, \tilde{E}_m \rangle \right)$$

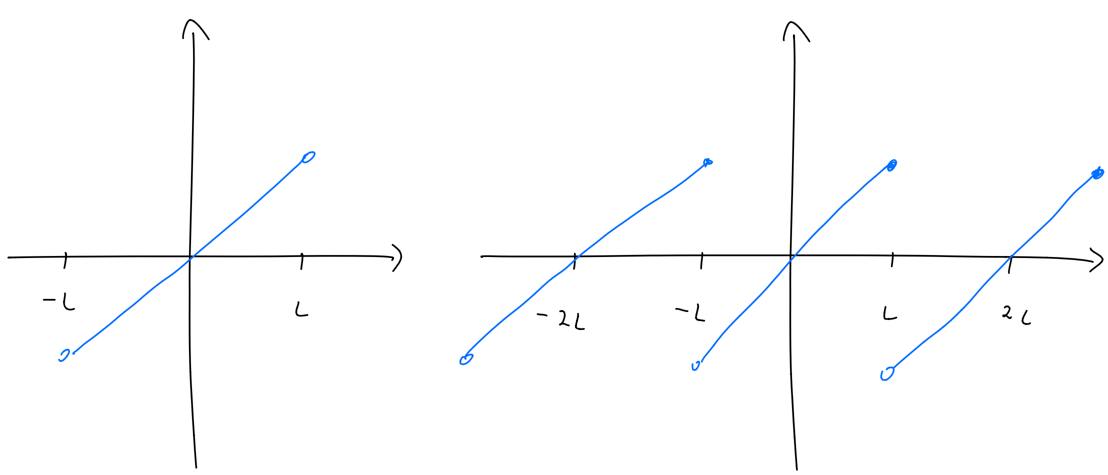
$$= b_m \langle \tilde{E}_m, \tilde{E}_m \rangle = b_m L \Rightarrow b_m = \frac{\langle f, \tilde{E}_m \rangle}{L}$$

Writing explicitly $\langle , \rangle$ in terms of an integral and using the definitions of $E_n, \tilde{E}_n$, we see that the expressions we found for a_n, b_n are exactly the Fourier coefficients.

The Fourier series of periodic functions, and the Fourier series of functions on $[0, L]$

Suppose that f is defined on $\mathbb{R}$ and has period $2L$, i.e., $f(x) = f(x+2L)$ for all x. Thus, all information about f is determined by its values on $[-L, L]$. We can define the Fourier series for f as a function on $[-L, L]$, and all the previous results are immediately adapted to this case.

Moreover, given a function on $(-L, L)$, we can extend it to a periodic function on $\mathbb{R}$ and consider its Fourier series (note, however, that this extension is not unique). This is illustrated in the picture below:



Consider now a function f defined on $[0, L]$. We define its cosine Fourier series by

$$F.S.^{\cos}\{f\}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi L}{x}\right), \quad x \in [0, L],$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

Extend f to an even function on $[-L, L]$ by

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(-x), & -L \le x < 0. \end{cases}$$

The Fourier coefficients of $\tilde{f}$ are

$$\tilde{a}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = a_n,$$

$$\tilde{b}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\left(\frac{n\pi x}{2}\right)dx = 0,$$

where we used that $\tilde{f}$ is even. Thus, for $x \in [0, L]$

$$F.S.\{\tilde{f}\}(x) = F.S.^{\cos}\{f\}(x).$$

In other words, the cosine Fourier series of $f: [0,L] \to \mathbb{R}$ equals the restriction to $[0,L]$ of the Fourier series of the even extension of f.

Similarly, we define the sine Fourier series of $f: [0,L] \to \mathbb{R}$ by

$$F.S.^{\sin}\{f\}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Letting $\tilde{f}$ be an odd extension of f,

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(-x), & -L \le x < 0, \end{cases}$$

we find the Fourier coefficients of $\tilde{f}$ to be

$$\tilde{a}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

$$\tilde{b}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = b_n,$$

thus $\quad F.S\{\tilde{f}\}(x) = F.S.^{\sin}\{f\}(x), \quad x \in [0,L].$

In other words, the sine Fourier series of $f:[0,L]\to\mathbb{R}$ equals the restriction to $[0,L]$ of the Fourier series of the odd extension of f.

We conclude that the theorems on convergence, differentiation, and integration of Fourier series are immediately applicable to the sine and cosine Fourier series.

Back to the wave equation.

We are now ready to discuss the problem

$$(*)\quad \begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } (0,\infty)\times(0,L), \quad c>0, \\ u(t,0) = u(t,L) = 0, & t \geq 0, \\ u(0,x) = g(x), & 0 \leq x \leq L, \\ \partial_t u(0,x) = h(x), & 0 \leq x \leq L, \end{cases}$$

where g and h are given functions satisfying the compatibility conditions

$$g(0) = g(L) = 0 = h(0) = h(L).$$

We saw that a formal solution to this problem is given by:

$$u(t,x) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right) \right) \sin\left(\frac{n\pi}{L} x\right) \quad (**)$$

where a_n and b_n are to be determined by

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} x\right),$$

and

$$h(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi}{L} x\right).$$

The last two expressions mean that g and h equal their sine Fourier series, with Fourier coefficients given by a_n and $\frac{n\pi c}{L} b_n$, respectively. These equalities will in fact be true if we make suitable assumptions on g and h. Let us assume that g and h are C^2 functions. Then, from the previous theorems for Fourier series, we know that g and h equal their sine Fourier series, and the coefficients a_n and b_n are given by

$$a_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{2}{n\pi c}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx. \quad (***)$$

Our assumptions on g and h allow us to compute the coefficients a_n and b_n. We will have to develop a few more tools before we are able to show that $(**)$ is in fact a solution. However, we summarize the result here; its proof will be postponed (in fact, it will be assigned as a HW after more background is developed).

Theo. Consider the problem $(*)$ and assume that g and h are C^2 functions such that

$$g(0) = g(L) = 0 = h(0) = h(L),$$

$$g''(0) = g''(L) = 0 = h''(0) = h''(L)$$

Then a solution to $(*)$ is given by $(**)$, where a_n and b_n are given by $(***)$.

Remark. We will explain the assumptions involving second derivatives of g and h when we prove this theorem.

The 1d wave equation in $\mathbb{R}$

We now consider the problem for $u = u(t,x)$:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in } (0,\infty)\times(-\infty,\infty), \quad c>0, \\ u(0,x) = u_0(x), \quad -\infty < x < \infty, \\ \partial_t u(0,x) = u_1(x), \quad -\infty < x < \infty. \end{cases}$$

This is an initial-value problem for the wave equation. Compared to the initial-boundary value problem we studied earlier, we see that now $x \in \mathbb{R}$, so there are no boundary conditions. This initial-value problem is also known as the Cauchy problem for the wave equation, a terminology that we will explain in more detail later on. We refer to the functions u_0 and u_1 as (initial) data for the Cauchy problem. A solution to this Cauchy problem is a function that satisfies the wave equation and the initial conditions.

We had defined the spaces $C^k(I)$ for an interval $I \subseteq \mathbb{R}$. For functions of two variables, we can similarly define $C^k(\mathbb{R}^2)$, which we will use here. We will define general C^k spaces for functions several variables later on.

Prop. Let $u \in C^2(\mathbb{R}^2)$ be a solution to the 1d wave equation. Then, there exist functions $F, G \in C^2(\mathbb{R})$ such that

$$u(t,x) = F(x+ct) + G(x-ct).$$

proof. Set $\alpha := x+ct$, $\beta := x-ct$, so that $t = \frac{1}{2c}(\alpha - \beta)$, $x = \frac{1}{2}(\alpha+\beta)$, and

$$v(\alpha,\beta) := u(t(\alpha,\beta), x(\alpha,\beta)).$$

Then, from $u(t,x) = v(\alpha(t,x), \beta(t,x))$ we find

$$u_t = v_\alpha \alpha_t + v_\beta \beta_t = c v_\alpha - c v_\beta,$$

$$u_{tt} = c v_{\alpha\alpha} \alpha_t + c v_{\alpha\beta} \beta_t - c v_{\beta\alpha} \alpha_t - c v_{\beta\beta} \beta_t$$

$$= c^2 v_{\alpha\alpha} - c^2 v_{\alpha\beta} - c^2 v_{\beta\alpha} + c^2 v_{\beta\beta},$$

$$u_x = v_\alpha \alpha_x + v_\beta \beta_x = v_\alpha + v_\beta,$$

$$u_{xx} = v_{\alpha\alpha}\alpha_x + v_{\alpha\beta}\beta_x + v_{\beta\alpha}\alpha_x + v_{\beta\beta}\beta_x$$

$$= v_{\alpha\alpha} + v_{\alpha\beta} + v_{\beta\alpha} + v_{\beta\beta}.$$

Thus, $0 = u_{tt} - c^2 u_{xx} = -4c^2 v_{\alpha\beta}$, where we used that $v_{\alpha\beta} = v_{\beta\alpha}$ since v is C^2 (because u is C^2 and the change of coordinates $(t,x) \mapsto (\alpha,\beta)$ is C^∞). Thus, in (α,β) coordinates the wave equation reads: $v_{\alpha\beta} = 0$.

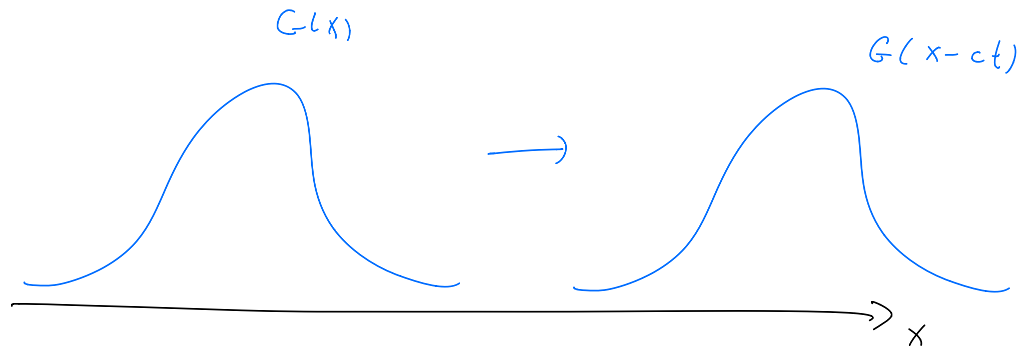
Therefore, $(v_\alpha)_\beta = 0$ implies that v_α is a function of α only: $v_\alpha(\alpha,\beta) = f(\alpha)$ for some C^1 function f. Integrating w.r.t. α gives

$$v(\alpha,\beta) = \int f(\alpha)\, d\alpha + G(\beta),$$

for some function G. Note that $F := \int f(\alpha)\, d\alpha$ is C^2, thus so is G. Therefore, $v(\alpha,\beta) = F(\alpha) + G(\beta)$, and in (t,x) coordinates:

$$u(t,x) = F(x+ct) + G(x-ct). \qquad \square$$

The above formula has a clear physical interpretation. At $t=0$, $u(0,x) = F(x) + G(x)$. For each $t>0$, the graph of $G(x-ct)$ is the graph of $G(x)$ moved ct units to the right, so the graph of $G(x)$ is moving to the right with speed c. $G(x-ct)$ is called a forward wave. Similarly, the graph of $F(x)$ is moving to the left and $F(x+ct)$ is called a backward wave. The general solution is thus a sum (or a superposition) of a forward and a backward wave, and we see that the constant c is indeed the speed of propagation of the wave.



Notation. Having found the interpretation of the constant c, we will often set $c = 1$.

Prop. Let $u \in C^2([0,\infty) \times \mathbb{R})$ be a solution to the Cauchy problem for the 1d wave equation with data u_0, u_1. Then

$$u(t,x) = \frac{u_0(t+x) + u_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy.$$

This formula is known as D'Alembert's formula.

proof. Note that $u_0 \in C^2$, $u_1 \in C^1$. From

$$u(t,x) = F(x+t) + G(x-t)$$

we have

$$u(0,x) = F(x) + G(x) = u_0(x),$$
$$u_t(0,x) = F'(x) - G'(x) = u_1(x).$$

Integrating this last equality:

$$F(x) - G(x) = \int_0^x u_1(y)\,dy + \overbrace{C}^{\text{constant} = F(0) - G(0)},$$

adding to $u(0,x)$:

$$F(x) = \frac{1}{2} u_0(x) + \frac{1}{2}\int_0^x u_1(y)\,dy + \frac{C}{2}.$$

Plugging back into $u(0,x)$:

$$G(x) = \frac{1}{2} u_0(x) - \frac{1}{2}\int_0^x u_1(y)\,dy - \frac{C}{2}.$$

Replacing $x \mapsto x+t$ in F and $x \mapsto x-t$ in G and adding gives the result. $\square$

The last two propositions derived formulas for C^2 solutions of the wave equation given such a solution. The next result shows that solutions actually exist:

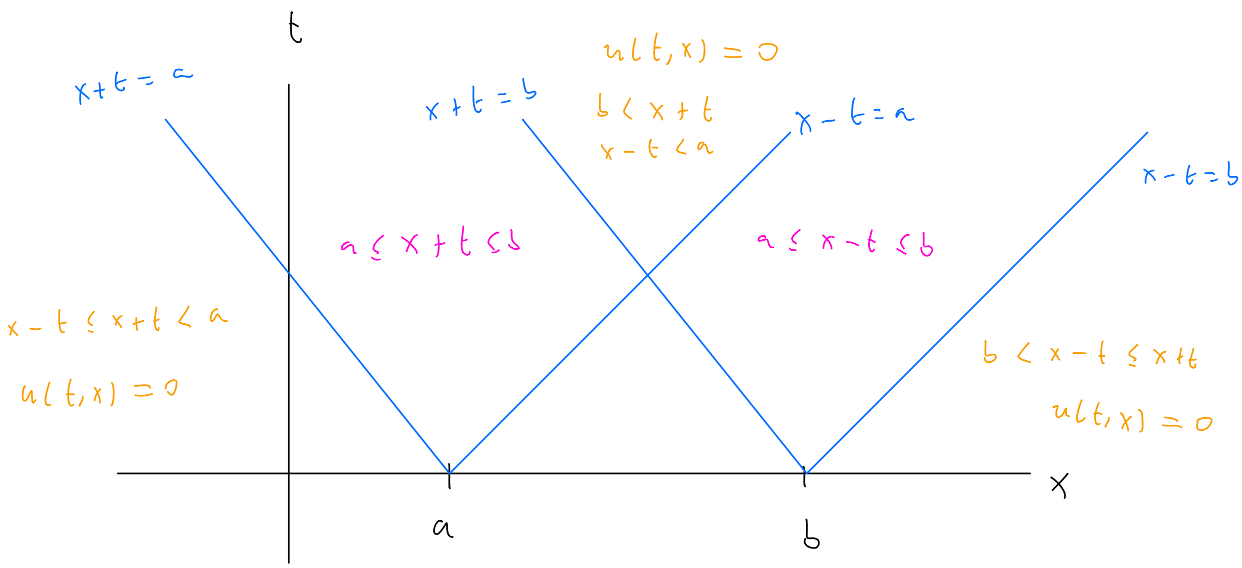
Theo. Let $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$. Then there exists a unique $u \in C^2([0,\infty) \times \mathbb{R})$ that solves the Cauchy problem for the wave equation with data u_0, u_1. Moreover, u is given by D'Alembert's formula.

proof. Given two $C^2([0,\infty) \times \mathbb{R})$ solutions, both satisfy D'Alembert's formula (with the same u_0, u_1) thus they are equal, establishing uniqueness. To prove existence, define u by D'Alembert's formula. Then $u \in C^2([0,\infty) \times \mathbb{R})$ since $u_0 \in C^2$ and $u_1 \in C^1$, and by construction (or direct computation) u satisfies the wave equation and the initial conditions. $\square$

Def. The lines $x + t = \text{constant}$ and $x - t = \text{constant}$ in the (t,x) plane (or $x + ct = \text{constant}$, $x - ct = \text{constant}$ for $c \neq 1$) are called the characteristics (or characteristic curves) of the wave equation. They (and their generalizations to higher dimensions) are very important to understand solutions to the wave equation, as we will see.

Regions of influence for the 1d wave equation

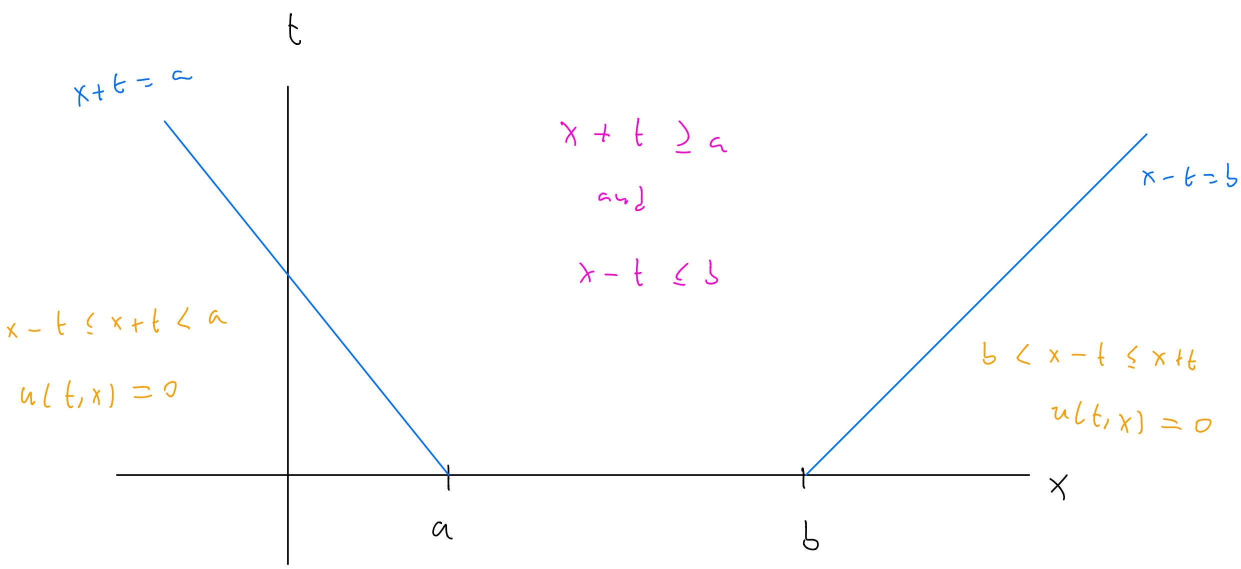
Suppose $u_1 = 0$ and $u_0(x) = 0$ for $x \notin [a,b]$. Since $u_0(x+t)$ and $u_0(x-t)$ are constant along the lines $x+t = \text{constant}$ and $x-t = \text{constant}$, respectively, we see that $u(t,x) \neq 0$ only possibly for points (t,x) that lie in the region determined by the region lying between the characteristics emanating from a and b as indicated in the figure:



<u>Notation.</u> Although we ordered the coordinates as (t,x), we will often draw the (t,x) plane with the x-axis on the horizontal.

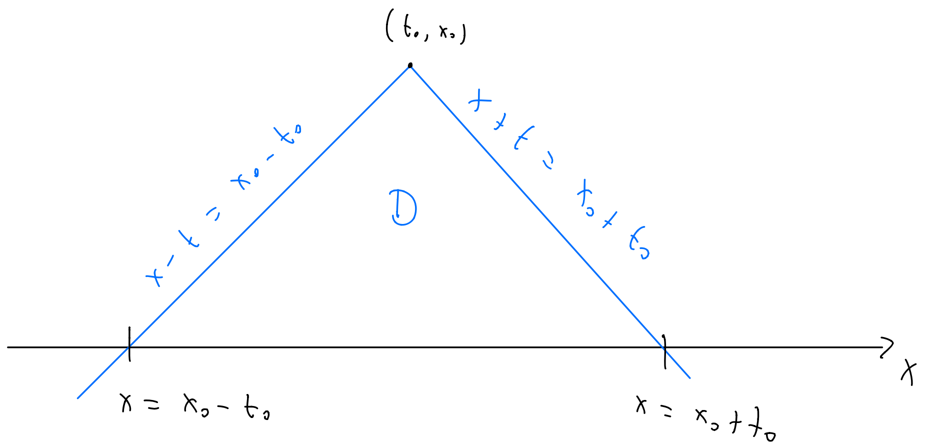
Suppose now that $u_0 = 0$ and that $u_1(x) = 0$ for

$x \notin [a,b]$. Then $\int_{x-t}^{x+t} u_1(y)\,dy = 0$ whenever we have $[x-t, x+t] \cap [a,b] = \emptyset$, i.e., if $x+t < a$ or $x-t > b$. Therefore, $u(t,x) \neq 0$ possibly only in the region $\{x+t \geq a\} \cap \{x-t \leq b\}$, as depicted in the figure



For general u_0 and u_1, we can therefore precisely track how the values of $u(t,x)$ are influenced by the values of the initial conditions. It follows that the values of the data on an interval $[a,b]$ can only affect the values of $u(t,x)$ for $(t,x) \in \{x+t \geq a\} \cap \{x-t \leq b\}$. This reflects the fact that waves travel at a finite speed. The region $\{x+t \geq a\} \cap \{x-t \leq b\}$ is called domain of influence of $[a,b]$.

Consider now a point (t_0, x_0) and $u(t_0, x_0)$. Let D be the triangle with vertex (t_0, x_0) determined by $x + t = x_0 + t_0$, $x - t = x_0 + t_0$, and $t = 0$:



then

$$u(t_0, x_0) = \frac{u_0(x_0 + t_0) + u_0(x_0 - t_0)}{2} + \frac{1}{2}\int_{x_0 - t_0}^{x_0 + t_0} u_1(y)\,dy$$

and we see that $u(t_0, x_0)$ is completely determined by the values of the initial data on the interval $[x_0 - t_0, x_0 + t_0]$. The region D is called the (past) domain of dependence of (t_0, x_0).

Generalized solutions.

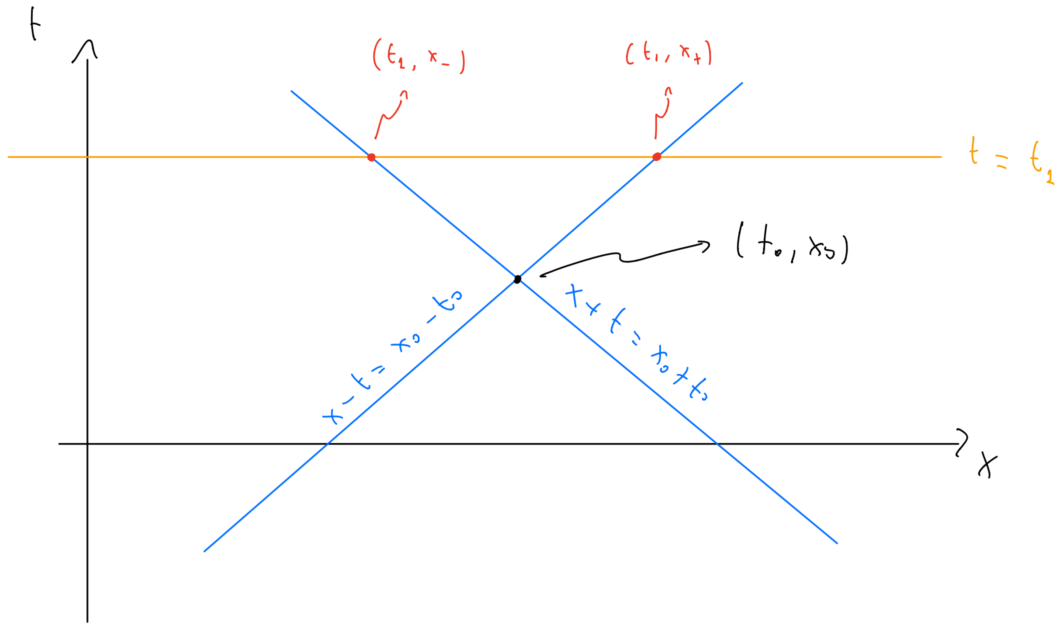
Note that the RHS of D'Alembert's formula makes sense when u_0 and u_1 are piecewise functions. This motivates the following definition.

Def. Let u_0 be a piecewise C^2 function and u_1 a piecewise C^1 function. Then u given by D'Alembert's formula is called a generalized solution to the wave equation. If u_0 and u_1 are C^2 and C^1 functions, respectively, then u is called a classical solution. When u is a generalized solution, the points where u fails to be C^2 are called the singularities of the solution (sometimes we abuse language and say singularities of the wave equation).

To understand what is going on, consider the case when for fixed t_0 u is C^2 except at the point (t_0, x_0). Writing $u(t,x) = F(x+t) + G(x-t)$, we see that F is not C^2 at x_0+t_0 and/or G is not C^2 at $x_0 - t_0$. The two characteristics passing through (t_0, x_0) are $x+t = x_0+t_0$ and $x-t = x_0 - t_0$.

Thus, for any fixed t_1, $u(t_1, x)$ fails to be C^2 except at one or two points, namely, $x_\pm$ such that

$$x_+ + t_1 = x_0 + t_0 \quad , \quad x_- - t = x_0 - t_0 .$$



This shows that the singularities of the wave equation remain localized in space and travel along the characteristics.

We will see that the results we obtained for the 1d wave equation (existence and uniqueness for the Cauchy problem, existence of domains of influence/dependence, propagation of singularities along characteristics) hold for the wave equation in higher dimensions and, in fact, for a class of equations called hyperbolic, of which the wave equation is the prototypical example.

Some general tools, definitions, and conventions for the study of PDEs

In order to advance further our study of PDEs, in particular to study PDEs in $\mathbb{R}^n$, we will recall a few tools from multivariable calculus and introduce some convenient notation/terminology.

Domains and boundaries

Def. A domain in $\mathbb{R}^n$ is an open connected subset of $\mathbb{R}^n$. If $\Omega \subseteq \mathbb{R}^n$ is a domain, we denote by $\overline{\Omega}$ its closure in $\mathbb{R}^n$. The boundary of a domain Ω, denoted $\partial\Omega$, is the set $\partial\Omega := \overline{\Omega} \setminus \Omega$. We say that a boundary $\partial\Omega$ has regularity C^k or is a C^k boundary if it can be written locally as the graph of a C^k function.

Notation. We denote by $|x|$ the Euclidean norm of an element $x \in \mathbb{R}^n$. Ω and $\partial\Omega$ will always denote a domain and its boundary, unless stated otherwise.

Ex: $B^n := \{ x \in \mathbb{R}^n \mid |x| < 1 \}$ is a domain in $\mathbb{R}^n$. Its boundary is the $n-1$ dimensional sphere:

$$S^{n-1} := \partial B^n = \{ x \in \mathbb{R}^n \mid |x| = 1 \}.$$

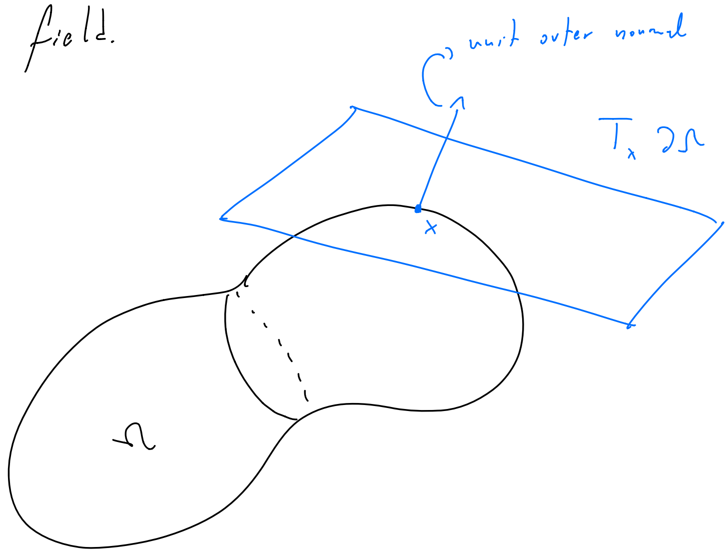
It is not difficult to see that S^{n-1} is C^∞, i.e., B^n has a C^∞ boundary. For example, the upper cap of S^{n-1}, given by $S^{n-1} \cap \{x^n > 0\}$, is the graph of the function $f : B^{n-1} \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ given by

$$f(x^1, \ldots, x^{n-1}) = \sqrt{1 - (x^1)^2 - \cdots - (x^{n-1})^2},$$

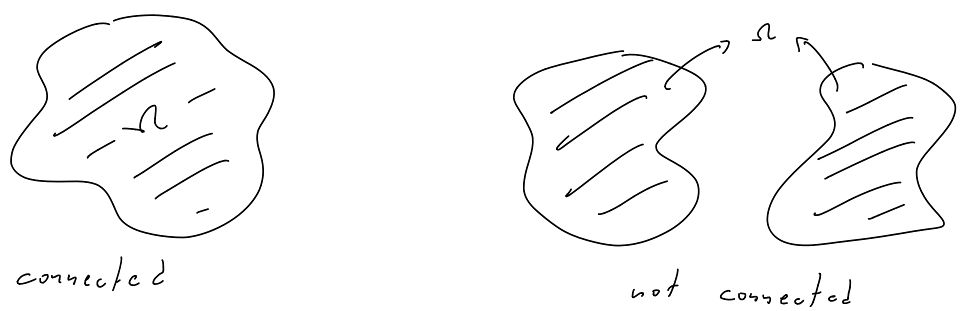
which is C^∞.

<u>Notation</u>. When talking about maps between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, we will often write $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where it is implicitly understood that the domain $\mathcal{U}$ of f is an open set (unless said otherwise).

Recall that if $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^1, for each $x \in \mathcal{U}$ the graph of f at $(x, f(x))$ admits a tangent plane. Thus, if $\partial\Omega$ is C^1, for each $x \in \partial\Omega$ there exists a <u>tangent plane</u> to $\partial\Omega$ at x, denoted $T_x\partial\Omega$. The <u>unit outer normal</u> to $\partial\Omega$ at x is by definition the unit normal to $T_x\partial\Omega$ that points to the exterior of Ω. The collection of unit outer normals N_x as x varies over $\partial\Omega$ forms a vector field over $\partial\Omega$ which is called the <u>unit outer normal vector field</u>. We sometimes refer simply to "the unit outer normal" when the context makes it clear whether we are talking about the vector field or a specific vector field.



<u>Remark.</u> Above, we took for granted that students recall (or have seen) the definition of a connected set in $\mathbb{R}^n$. Intuitively, a set is connected if it is not "split into separate parts:"



For the time being, this intuitive notion will suffice for students who have not seen the precise definition. The mathematical definition of connectedness will be given later on.

The Kronecker delta

Def. The Kronecker delta symbol in n dimensions, or simply the Kronecker delta when the dimension is implicitly understood, is defined as the collection of numbers $\{\delta_{ij}\}_{i,j=1}^{n}$ such that $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij} = 0$ if $i \neq j$. We identify the Kronecker delta with the entries of the $n \times n$ identity matrix in standard coordinates. We also define $\delta^{ij} := \delta_{ij}$, which we also call the Kronecker delta and identify with the entries of the identity matrix.

Recall that the Euclidean inner product, a.k.a. the dot product, of vectors in $\mathbb{R}^n$ is the map:

$$\langle\, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

given in standard coordinates by:

$$\langle X, Y \rangle = \sum_{i=1}^{n} X^i Y^i ,$$

which is also denoted by $X \cdot Y$. We can write $\langle X, Y \rangle$ as (recall our sum convention):

$$\langle X, Y \rangle = \delta_{ij} X^i Y^j .$$

In view of this last formula, we also identify the Kronecker delta with the Euclidean inner product.

Raising and lowering indices with δ

Given a vector $\mathbb{X} = (\mathbb{X}^1, \dots, \mathbb{X}^n)$, we define

$$\mathbb{X}_i := \delta_{ij} \mathbb{X}^j, \quad i = 1, \dots, n.$$

We say that we are lowering the indices of $\mathbb{X}$ and identify the n-tuple $(\mathbb{X}_1, \dots, \mathbb{X}_n)$ with the vector $\mathbb{X}$ itself.

The point of introducing $\mathbb{X}_i$ is to achieve consistency with our convention of summing indices that appear once up and once down. For example, if we want the inner product

$$\langle \mathbb{X}, \mathbb{Y} \rangle = \sum_{i=1}^n \mathbb{X}^i \mathbb{Y}^i$$

using our sum convention (thus avoiding to write $\sum_{i=1}^n$), one of the indices i needs to be downstairs:

$$\langle \mathbb{X}, \mathbb{Y} \rangle = \mathbb{X}^i \mathbb{Y}_i,$$

so that we had to break with our convention that vectors have indices upstairs. However, if we now interpret Y_i as lowering the indices of Y, then

$$\langle X, Y \rangle = \delta_{ij} X^i Y^j = X^i \underbrace{\delta_{ij} Y^j}_{= Y_i} = X^i Y_i .$$

Similarly, recall that we wrote:

$$\operatorname{curl}^i X = \varepsilon^{ijk} \partial_j X_k ,$$

where we had artificially written X_k with an index downstairs, thus breaking with our convention that vectors had an index upstairs. But now we have a proper way of thinking of X_k as $\delta_{kj} X^j$.

Note that using δ_{ij} we could completely avoid writing vectors with indices downstairs, i.e., every time that X_i appears in a formula we can replace it with $\delta_{ij} X^j$. E.g.,

$$\mathrm{curl}^i X = \varepsilon^{ijk} \delta_{kl} \partial_j X^l .$$

But the point is precisely to have a compact notation, so $\delta_{kl} \partial_j X^l = \partial_j \delta_{kl} X^l = \partial_j X_k$.

<u>Remark.</u> In the above computations, note that we can move δ_{kl} pass the derivative because δ_{kl} is constant for each fixed k and l, i.e., δ_{kl} is not a function of the coordinates.

We extend the lowering of indices to any object indexed by $i_1, \dots, i_l$, $i_j \in \{1, \dots, n\}$, $j = 1, \dots, n$. E.g.:

$$\varepsilon_i{}^{jk} := \delta_{il} \varepsilon^{ljk},$$

$$\varepsilon^i{}_j{}^k := \delta_{jl} \varepsilon^{ilk}, \quad \text{etc.}$$

Note that it is important to keep the order of the indices on the LHS due to the anti-symmetry of ε, so that $\varepsilon^i{}_j{}^k \neq \varepsilon_j{}^{ik}$. In fact, the order of the indices always matters unless one is dealing with objects that are symmetric in the respective indices. E.g., if a^{ij} are the entries of a matrix, then

$$a_i{}^j := \delta_{i\ell}a^{\ell j}$$

and in general $a_i{}^j \neq a^j{}_i$. However, if the matrix is symmetric, $a^{ij} = a^{ji}$, then $a_i{}^j = a^j{}_i$ and we write a^j_i for $a_i{}^j$.

The same way we lowered indices using δ_{ij}, we can raise indices using δ^{ij}. For instance, given an object indexed by downstairs indices ij, i.e., A_{ij}, we set

$$A^i{}_j := \delta^{i\ell}A_{\ell j}.$$

Again, the order of the indices on the LHS matters unless the object is symmetric. It follows that we can define the Kronecker delta with one index up and one down:

$$\delta^i_{\ j} = \delta^{il}\delta_{lj}.$$

It follows that

$$\delta^i_{\ j} = \begin{cases} 1, & i=j, \\ 0, & i\neq j. \end{cases}$$

Note that raising and then lowering (or vice-versa) an index gives the same object back. E.g.:

$$\cancel{X}_i = \delta_{ij}\cancel{X}^j \Rightarrow \cancel{X}^i = \delta^{ij}\cancel{X}_j = \underbrace{\delta^{ij}\delta_{jl}}_{=\delta^i_{\ l}}\cancel{X}^l = \cancel{X}^i,$$

where we used $\delta^i_{\ l} = 0$ for $i \neq l$.

Recall that $\partial_i = \frac{\partial}{\partial x^i}$. We define the derivative with an index upstairs by:

$$\partial^i := \delta^{ij}\partial_j.$$

Using this notation, we can write the Laplacian as:

$$\Delta = \partial^i\partial_i = \delta^{ij}\partial_i\partial_j.$$

We sometimes abbreviate $\partial^2_{ij} = \partial_i\partial_j$, $\partial^3_{ijk} = \partial_i\partial_j\partial_k$, etc.

Important remark. The use of the Kronecker delta and the raising and lowering of indices provide us with a convenient and compact notation. But the overall discussion and definitions probably seem a bit ad hoc. It turns out that these ideas can be given a more satisfactory content within the language of differential geometry. For example, the Kronecker delta can be introduced not as a "collection of symbols" but rather as a tensor satisfying certain properties. The raising and lowering of indices can be interpreted as a map, given by the inner product, that identifies elements of a vector space and its dual, or vector fields and one forms; or yet more generally as the identification of covariant and contravariant tensors. Since we will not be discussing differential geometry (except for some elementary aspects tied to PDEs), here we will take a purely instrumental point of view, using the above machinery mostly as a matter of convenient notation.

Calculus facts

We colect a few calculus facts that we will use later on.

Def. We say that a map f is k-times continuously differentiable if all its partial derivatives up to order k exist and are continuous in the domain of f. We denote the space of k-times continuously differentiable functions in $\mathcal{U} \subseteq \mathbb{R}^n$ by $C^k(\mathcal{U})$. Sometimes we write simply C^k if $\mathcal{U}$ is implicitly understood, and sometimes we say simply "f is C^k" to mean that f is k-times continuously differentiable.

Integration by parts. If $u, v \in C^1(\bar{\Omega})$, then

$$\int_\Omega \partial_i u \, v \, dx = -\int_\Omega u \, \partial_i v \, dx + \int_{\partial\Omega} u v \nu^i \, dS,$$

$i = 1, \ldots, n$, where $\nu = (\nu^1, \ldots, \nu^n)$ is the unit outer normal to $\partial\Omega$ and dS is the volume element induced on $\partial\Omega$.

Students who have not seen the above integration by parts in $\mathbb{R}^n$ can view it as a generalization of the divergence theorem in $\mathbb{R}^3$. The latter can be written (using Stewart's Calculus notation):

$$\iiint_E \operatorname{div} \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{S}.$$

Take $\vec{F} = uv\vec{e}_i$, where $\vec{e}_i$ has 1 in the ith component and zero in the remaining components. Then,

$$\operatorname{div} \vec{F} = \partial_i u v + u \partial_i v.$$

For example, if $\vec{e}_i = e_1 = (1, 0, 0)$, and writing $\vec{F} = (F_x, F_y, F_z)$, so that

$$\operatorname{div} \vec{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z,$$

we find

$$\operatorname{div} \vec{F} = \operatorname{div} (uv, 0, 0) = \partial_x (uv)$$

$$= \partial_x u v + u \partial_x v,$$

and similarly for $\vec{e}_2$ and $\vec{e}_3$. Recalling also that $d\vec{S} = \vec{n}\, dS$, where $\vec{n}$ is the unit outer normal,

$$\vec{F} \cdot d\vec{S} = (u v \vec{e}_i) \cdot \vec{n}\, dS = u v\, \vec{e}_i \cdot \vec{n}\, dS.$$

But $\vec{e}_i \cdot \vec{n} = i^{th}$ component of $\vec{n} = n^i$, thus

$$\vec{F} \cdot d\vec{S} = u v n^i.$$

Plugging the above into the divergence theorem:

$$\iiint_E (\partial_i u v + u \partial_i v)\, dV = \iint_S u v n^i\, dS$$

which is the formula we stated in a different notation.

<u>Def</u>. Let $u \in C^1(\bar{\Omega})$. The <u>normal derivative</u> of u, denoted $\frac{\partial u}{\partial \nu}$, is a function defined on $\partial\Omega$ by

$$\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu,$$

where ν is the unit outer normal to $\partial\Omega$ and ∇ is the gradient.

From the integration by parts formula we can derive the following formulas (sometimes called Green's identities):

For $u \in C^1(\bar{\Omega})$:

$$\int_{\Omega} \partial_i u \, dx = \int_{\partial\Omega} u \nu^i \, dS .$$

For $u, v \in C^2(\bar{\Omega})$:

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS ,$$

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = - \int_{\Omega} u \Delta v + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, dS ,$$

$$\int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS$$

Formal aspects of PDEs

Def. and notation. A vector of the form
$$\alpha = (\alpha_1, \dots, \alpha_n),$$
where each entry is a non-negative integer is called a multiindex of order $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Given a multiindex, we define:
$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial (x^1)^{\alpha_1} \dots \partial (x^n)^{\alpha_n}},$$
where $u = u(x^1, \dots, x^n)$. If k is a non-negative integer,
$$D^k u := \{ D^\alpha u \mid |\alpha| = k \}$$
is the set of all k^{th} order partial derivatives of u. When $k=1$ we identify Du with the gradient of u. When $k=2$ we identify $D^2 u$ with the Hessian matrix of u:
$$D^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial (x^1)^2} & \dots & \frac{\partial^2 u}{\partial x^1 \partial x^n} \\ \vdots & & \\ \frac{\partial^2 u}{\partial x^n \partial x^1} & \dots & \frac{\partial^2 u}{\partial (x^n)^2} \end{bmatrix}$$

We can regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$. Its norm is

$$|D^k u(x)| = \sqrt{\sum_{|\alpha|=k} |D^\alpha u(x)|^2},$$

where $\sum\limits_{|\alpha|\le k}$ means the sum is over all multiindices of order k.

If $u = (u^1, \dots, u^m)$ is vector valued, we define

$$D^\alpha u := (D^\alpha u^1, \dots, D^\alpha u^m)$$

and set

$$D^k u := \{ D^\alpha u \mid |\alpha| = k \},$$

and

$$|D^k u| = \sqrt{\sum_{|\alpha|=k} |D^\alpha u|^2}$$

as before.

We will now restate the definition of PDEs using the above notation. This new definition agrees with the one previously given.

Def. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $k \geq 1$ be a non-negative integer. An expression of the form

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0,$$

$x \in \Omega$, is called a k^{th} order partial differential equation (PDE), where:

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is the unknown. A solution to the PDE is a function u that verifies the PDE. Sometimes we drop x from the notation and state the PDE as

$$F(D^k u, D^{k-1} u, \ldots, Du, u, x) = 0 \text{ in } \Omega.$$

Ω is sometimes called the domain of definition of the PDE.

Ex: $\Delta u = 0$ in $\mathbb{R}^3$ can be written as

$$F(D^2 u, Du, u, x) = 0 \text{ in } \mathbb{R}^3$$

with $F : \mathbb{R}^9 \times \mathbb{R}^3 \times \mathbb{R} \times \underbrace{\mathbb{R}^3}_{=\Omega} \to \mathbb{R}$ given by the following expression. First, we label the coordinates in $\mathbb{R}^9 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ according to the order of the derivatives and x, i.e.,

$$\frac{\partial^2 u}{\partial (x^1)^2}, \frac{\partial^2 u}{\partial x^1 \partial x^2}, \dots, \frac{\partial^2 u}{\partial (x^3)^2}, \frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^3}, u, x^1, x^2, x^3,$$

so

$$F = F(\underbrace{p_{11}, p_{12}, p_{13}, p_{21}, \dots, p_{33}}_{9 \text{ entries}}, \overbrace{p_1, p_2, p_3}^{3 \text{ entries}}, p, x^1, x^2, x^3).$$

Then F is given by

$$F(p_{11}, \dots, x^3) = p_{11} + p_{22} + p_{33}.$$

<u>Ex</u>: $\Delta u = f$ in $\mathbb{R}^3$, where $f(x) = (x^1)^2 + (x^2)^2 + (x^3)^2$, can be written, using the notation of the previous example, as in the definition with F given by

$$F(p_{11}, \dots, x^3) = p_{11} + p_{22} + p_{33} - ((x^1)^2 + (x^2)^2 + (x^3)^2).$$

Def. A PDE

$$F(D^k u, D^{k-1}u, \dots, Du, u, x) = 0$$

is called linear if F is linear in all its entries except possibly in x. Otherwise it is called non-linear. More precisely, denoting $F: \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$,

by $F = F(\vec{p}, x)$

$$\vec{p} = (\underbrace{p_{k,1}, \dots, p_{k,n^k}}_{n^k \text{ entries for } \mathbb{R}^{n^k}}, \underbrace{p_{k-1,1}, \dots, p_{k-1,n^{k-1}}}_{n^{k-1} \text{ entries for } \mathbb{R}^{n^{k-1}}}, \dots, p)$$

we can write $F(\vec{p}, x) = F_H(\vec{p}, x) + F_I(x)$, where F_I contains all terms that do not on $\vec{p}$ (i.e., terms that do not depend on u or its derivatives). F_H is called the homogeneous part of F and F_I the inhomogeneous part. Thus, the PDE is called linear if F_H is linear in all its entries. The PDE is called homogeneous if $F_I \equiv 0$ an inhomogeneous otherwise.

We clarify that when we say that F_H is linear in, say, the entry $D^h u$, we mean that it is linear in each component of $D^h u$ separately. For instance, $F_H(Du, u, x)$ is linear if it is in particular linear in Du. Since $Du = (\partial_1 u, \dots, \partial_n u)$ we mean that F is linear in each entry of $(\partial_1 u, \dots, \partial_n u)$ plus in the entry u.
In other words, labeling the entries of $F = F(p_1, \dots, p_n, p, x^1, \dots, x^n)$, the PDE is linear if F is linear in each p_i, $i = 1, \dots, n$, and in p.

A linear PDE $F(D^h u, \dots, u, x)$ can always be written as

$$\sum_{|\alpha| \leq h} a_\alpha D^\alpha u = f ,$$

where the a_α and f are known functions defined on Ω.
If the PDE is also homogeneous then $f \equiv 0$.

A PDE as defined above, where the unknown is a single function on Ω, is also called a <u>scalar PDE</u>.

Def. A k^{th} order PDE is called semi-linear if it has the form

$$\sum_{|\alpha|=k} a_\alpha D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0,$$

where the $a_\alpha : \Omega \to \mathbb{R}$ and $a_0 : \mathbb{R}^{n^{k-1}} \times \dots \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ are given functions.

A k^{th} order PDE is called quasi-linear if it has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0,$$

where $a_\alpha, a_0 : \mathbb{R}^{n^{k-1}} \times \dots \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ are known functions.

A PDE is called fully non-linear if it depends non-linearly on its highest order derivative.

Def. An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du, u, x) = 0,$$

is called a k^{th} order system of PDEs, where

$$F = (F^1, \dots, F^\ell) : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \dots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^\ell$$

is given and

$$u = (u^1, \ldots, u^m) : \Omega \to \mathbb{R}^m$$

is the unknown. A solution to the system of PDEs is a function $u : \Omega \to \mathbb{R}^m$ that satisfies the system of PDEs. We sometimes drop the x-dependence and write

$$F(D^k u, \ldots, Du, u, x) = 0 \text{ in } \Omega.$$

We sometimes refer to a system of PDEs simply as a PDE.

The definitions of (non-) linear, (non-) homogeneous, semilinear and quasilinear generalize in a straightforward fashion to systems. In particular, A linear system can be written as

$$\sum_{|\alpha| \leq k} A_\alpha D^\alpha u = f,$$

where $A_\alpha : \Omega \to \mathbb{R}^{lm}$ are known $l \times m$ matrices (depending on $x \in \Omega$) and $f : \Omega \to \mathbb{R}^l$ is a known function ($f = 0$ if the system is homogeneous).

Having introduced the basic definitions and terminology for PDEs, let us discuss the case of evolution equations, i.e., when of the variables represents time.

When we study a PDE where one of the variables is the time variable, it is convenient to separate to separate time and space and denote the spatial variables by $(x^1, \dots, x^n)$ and the time variable by x^0. In this case we have $n+1$ variables and extend the multiindex notation to

$$\alpha = (\alpha_0, \dots, \alpha_n), \quad |\alpha| = \alpha_0 + \dots + \alpha_n,$$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial (x^0)^{\alpha_0} \partial (x^1)^{\alpha_1} \dots \partial (x^n)^{\alpha_n}}.$$

The domain of definition of the PDE in this case is $\Omega \subseteq \mathbb{R}^{n+1}$, but it is convenient to take it to be $(T_I, T_F) \times \Omega \subseteq \mathbb{R}^{n+1}$, for some interval $(T_I, T_F) \subseteq \mathbb{R}$ and some domain $\Omega \subseteq \mathbb{R}^n$. Typically $(T_I, T_F) = (0, T)$ for some $T > 0$. We also write $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ when we want to emphasize that the first coordinate, x^0, corresponds to time. We also write

$$t := x^0$$

for the time variable. Thus $\frac{\partial}{\partial t} = \frac{\partial}{\partial x^0}$

Notation. We extend our indices convention by adopting the convention that Latin lower-case indices range from 1 to n (as we have used so far) and Greek lower-case indices range from 0 to n. For instance,

$$
\begin{aligned}
a^\alpha \partial_\alpha u &= a^0 \partial_0 u + a^i \partial_i u \\
&= a^0 \partial_t u + a^i \partial_i u \\
&= a^0 \partial_t u + a^1 \partial_1 u + \dots + a^n \partial_n u.
\end{aligned}
$$

Note that we use Greek letters to denote both indices varying from 0 to n and multi-indices. The context will make the distinction clear. In particular, note that for multiindices we never use the convention that repeated indices are summed. Thus, for example, in $a^\alpha \partial_\alpha$, α is an index summed from 0 to n, whereas in $\sum_{|\alpha| \leq k} a^\alpha D^\alpha$, α is a multiindex summed over all multiindices with $|\alpha| \leq k$. Finally, if

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$$

is a multiindex, we write $\vec{\alpha}$ for its "spatial part," i.e.,

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$$

We next state some useful calculus facts using multiindex notation. The formulas below involve functions $u = u(x^1, \dots, x^n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, but clearly similar formulas hold for $u = u(x^0, x^1, \dots, x^n)$ and $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$. For multiindices α and β, we define $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$, $\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \ \forall\, i = 1, \dots, n$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Multinomial theorem:

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} x^\alpha,$$

where $\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}$.

Leibniz's formula or product rule:

$$D^\alpha (uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \, D^{\alpha - \beta} v,$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta! (\alpha - \beta)!}$.

Taylor's formula:

$$u(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha u(0) x^\alpha + \mathcal{O}(|x|^{k+1}) \quad \text{as } x \to 0.$$

Above, $u, v : \mathbb{R}^n \to \mathbb{R}$ are sufficiently regular as to make the formulas valid.

Remark. When we introduce a PDE, we indicate the domain Ω where it is defined, which says that we are looking for a solution that is defined in Ω. It may happen, however (and it is often the case for non-linear PDEs) that we are able to find a solution u, but u is defined only on a smaller domain $\Omega' \subset \Omega$. I.e, u satisfies the PDE only for $x \in \Omega'$, where Ω' is strictly smaller than Ω. In fact, we a priori do not know whether it is possible to satisfy the PDE for all $x \in \Omega$. We still call such a u that is defined only on Ω' a solution, and sometimes call it a local solution if we want to emphasize that the solution we found is defined on a domain smaller than where the PDE was originally stated. In other words, the domain of definition of the PDE is a guide that helps us define the problem, but it can happen that solutions are only defined in a subset of Ω.

Let us illustrate this situation with a simple ODE example. Consider

$$\frac{dy}{dt} = y^2 \quad \text{in } \Omega = (0, \infty), \text{ with initial condition } y(0) = 1.$$

The solution is $y(t) = \dfrac{1}{1-t}$. This solution, however, is not defined for $t = 1$. Thus we in fact have a local solution defined on

$\Omega' = (0,1)$ (we do not take $\Omega' = (0,1) \cup (1,\infty)$ because this set is not connected; and we take the portion $(0,1)$ because we need to approach zero to satisfy the initial condition).

We can also define boundary value problems, initial value problems, and initial-boundary value problems as we had done for the 1d wave equation. We will not give these general definition here, but will introduce them as needed to study specific problems. We note that in such cases we will in general seek a solution defined on a larger domain than Ω. For example, we may want $u: \bar{\Omega} \to \mathbb{R}$ in a boundary value problem or $u: [0,T) \times \Omega \to \mathbb{R}$ in an initial-value problem. What exactly is required is usually a case-by-case analysis.

Important notation on constants. In what follows we are going to derive estimates and computations that involve numerical constants whose specific value will not be important. Thus, we will denote by $C > 0$ a generic positive constant that can vary from line to line. C will generally depend on fixed data of the problem (e.g., the dimension n). Sometimes we indicate the dependence of C using subscripts, e.g., C_n.

Laplace's equation in $\mathbb{R}^n$

We are going to study Laplace's equation in $\mathbb{R}^n$:

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n,$$

and its inhomogeneous version known as Poisson's equation:

$$\Delta u = f \quad \text{in } \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given.

We begin looking for a solution of the form

$$u(x) = v(r),$$

where $r = |x| = ((x^1)^2 + \dots + (x^n)^2)^{1/2}$ is the distance to the origin. The motivation to look for such a solution is that Laplace's equation is rotationally invariant (this will be a HW). Direct computation gives:

$$\partial_i r = \frac{x^i}{r}, \quad x \neq 0,$$

$$\partial_i u = v' \frac{x^i}{r},$$

$$\partial_i^2 u = v'' \frac{(x^i)^2}{r^2} + v' \left(\frac{1}{r} - \frac{(x^i)^2}{r^3} \right).$$

Summing from 1 to n:

$$\Delta u = v'' + \frac{n-1}{r} v'.$$

Hence

$$\Delta u = 0$$

iff

$$v'' + \frac{n-1}{r} v' = 0,$$

which is a ODE for v (recall $v = v(r)$). If $v' \neq 0$ we can write it as

$$\left(\ln |v'|\right)' = \frac{v''}{v'} = \frac{1-n}{r},$$

which gives

$$v'(r) = \frac{A}{r^{n-1}},$$

for some constant A. If $r > 0$, integrating again we find

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \dfrac{a}{r^{n-2}} + b, & n \geq 3, \end{cases}$$

where a and b are arbitrary constants.

This calculation motivates the following definition.

Def. The function

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \ln|x|, & n = 2, \\ \frac{1}{n(2-n)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

is called the fundamental solution of Laplace's equation.

Above and henceforth, we adopt the following:

Notation. We denote by $B_r(x)$ the (open) ball of radius r centered at x in $\mathbb{R}^n$, i.e.,

$$B_r(x) := \{ y \in \mathbb{R}^n \mid |x-y| < r \}.$$

Sometimes we write $B_r^n(x)$ to emphasize the dimension. We denote:

$$\omega_n := \text{volume}(B_1^n(0)).$$

In particular $\omega_3 = \frac{4}{3}\pi$.

Note that $\Delta \Gamma(x) = 0$ for $x \neq 0$ by construction. Sometimes we write $\Gamma(|x|)$ to emphasize the radial dependence on $r = |x|$.

Before solving Laplace's equation, we need one more definition.

Def. The support of a map $f: U \to \mathbb{R}$ is the set

$$\operatorname{supp}(f) := \overline{\{ x \in U \mid f(x) \neq 0 \}},$$

where $\overline{}$ is the closure. Recall that a set $U \subseteq \mathbb{R}^n$ is called compact if it is closed and bounded. We say that f has compact support if $\operatorname{supp}(f)$ is compact. We denote by $C^k_c(U)$ the space of C^k functions in U with compact support.

Theo. Let $f \in C^2_c(\mathbb{R}^n)$. Set:

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy.$$

Then:

(i) u is well-defined.

(ii) $u \in C^2(\mathbb{R}^n)$

(iii) $\Delta u = f$ in $\mathbb{R}^n$.

proof: We will carry out the proof for $n \geq 3$. The case $n = 2$ is done with similar arguments.

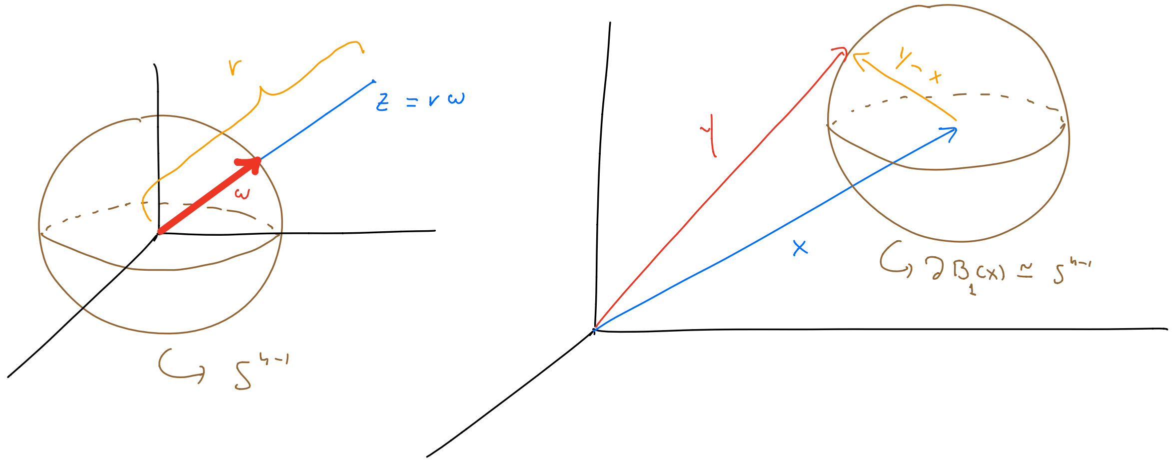
To begin, recall that a continuous function over a compact set always has a maximum and a minimum. Therefore, since f has compact support, there exists a constant $C' > 0$ such that $|f(x)| \leq C'$ for every x. Moreover, again by the compact support of f, there exists a $R > 0$ such that

$$\int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_{B_R(x)} \Gamma(x-y) f(y)\, dy.$$

Thus:

$$\left| \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy \right| \leq C' \int_{B_R(x)} |\Gamma(x-y)|\, dy \leq C \int_{B_R(x)} \frac{1}{|x-y|^{n-2}}\, dy.$$

We now take polar coordinates (r, ω) centered at x, where r = distance to x and $\omega \in S^{n-1}$ = $n-1$ dimensional unit sphere, so that $y - x = r\omega$, $|x-y| = r$.



In these coordinates $dy = r^{n-1} d\omega$, where $d\omega$ is the volume element on S^{n-1} (for $n=3$, $d\omega = \sin\phi\, d\phi\, d\theta$). Then

$$\int_{B_R(x)} \frac{1}{|x-y|^{n-2}}\,dy = \int_0^R \int_{S^{n-1}} \frac{1}{r^{n-2}} r^{n-1}\,dr\,d\omega = \int_0^R r\,dr \int_{S^{n-1}} d\omega = C,$$

showing that u is well defined, i.e., (i).

To prove (ii), first make a change of variables $z = x-y$, so

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\,dy = \int_{\mathbb{R}^n} \Gamma(z) f(x-z)\,dz.$$

Note that $\partial_i f$ and $\partial_{ij}^2 f$ also have compact support, thus an argument similar to the above shows that

$$\int_{\mathbb{R}^n} |\Gamma(y)\,\partial_i f(x-y)|\,dy \quad \text{and} \quad \int_{\mathbb{R}^n} |\Gamma(y)\,\partial_{ij}^2 f(x-y)|\,dy$$

are well defined. Let $e_i = (0,\dots,\underset{\text{ith entry}}{1},\dots,0)$ be the canonical basis vectors in $\mathbb{R}^n$ and let $h>0$. Then, for any x:

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Gamma(y)\left(\frac{f(x+he_i-y)-f(x-y)}{h}\right) dy.$$

$$= \int_{B_R(0)} \Gamma(y)\left(\frac{f(x+he_i-y)-f(x-y)}{h}\right) dy,$$

where the second equality holds for a sufficiently large R in view of the compact support of f.

Since $\lim_{h\to 0} \dfrac{f(x+e_ih-y)-f(x-y)}{h} = \partial_i f(x-y)$ and

the integral of $\Gamma(y)\partial_i f(x-y)$ is well defined,

$$\lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h} = \lim_{h\to 0} \int_{\mathbb{R}^n} \Gamma(y)\left(\frac{f(x+e_ih-y)-f(x-y)}{h}\right)dy$$

$$= \int_{\mathbb{R}^n} \Gamma(y)\left(\lim_{h\to 0}\frac{f(x+he_i-y)-f(x-y)}{h}\right)dy = \int_{\mathbb{R}^n} \Gamma(y)\,\partial_i f(x-y)\,dy,$$

showing that the limit $\lim_{h\to 0}\dfrac{u(x+he_i)-u(x)}{h}$ exists, i.e.,

$\partial_i u(x)$ exists. Repeating the argument with $f(x-y)$ replaced by $\partial_i f(x-y)$ we conclude that $\partial^2_{ij}u(x)$ exists and

$$\partial^2_{ij}u(x) = \int_{\mathbb{R}^n} \Gamma(y)\,\partial^2_{ij}f(x-y)\,dy.$$

To show that $u\in C^2(\mathbb{R}^n)$, it remains to show that $\partial^2_{ij}u$ is continuous. Fix $x_0\in\mathbb{R}^n$, fix $\varepsilon>0$, and consider:

$$|\partial^2_{ij}u(x_0)-\partial^2_{ij}u(x)| = \left|\int_{\mathbb{R}^n}\Gamma(y)\left(\partial^2_{ij}f(x_0-y)-\partial^2_{ij}f(x-y)\right)dy\right|$$

$$\leq \int_{\mathbb{R}^n}|\Gamma(y)|\,|\partial^2_{ij}f(x_0-y)-\partial^2_{ij}f(x-y)|\,dy.$$

Since $\partial_{ij}^2 f$ is continuous and has compact support it is uniformly continuous, i.e., given ε', there exists a $\delta > 0$ such that $|\partial_{ij}^2 f(z) - \partial_{ij}^2 (y)| < \varepsilon'$ whenever $|z-y| < \delta$. Putting $\varepsilon' = \frac{\varepsilon}{G'}$, with $G' = \int_{B_R(0)} |\Gamma(y)|\, dy$ (which we already know to be finite), we find that if $|x_0 - x| < \delta$, so that $|(x_0 - y) - (x-y)| < \delta$, we obtain that

$$|\partial_{ij}^2 u(x_0) - \partial_{ij}^2 u(x)| \leq \int_{B_R(0)} |\Gamma(y)| \underbrace{|\partial_{ij}^2 f(x_0 - y) - \partial_{ij}^2 f(x-y)|}_{< \varepsilon'} dy < \varepsilon,$$

showing that $u \in C^2(\mathbb{R}^n)$.

To show (iii), from the expression for $\partial_{ij} u$ we obtain

$$\Delta u(x) = \delta^{ij} \partial_{ij}^2 u(x) = \int_{\mathbb{R}^n} \Gamma(y) \, \Delta_x f(x-y) \, dy,$$

$$= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \, \Delta_x f(x-y) \, dy + \int_{B_\varepsilon(0)} \Gamma(y) \, \Delta_x f(x-y) \, dy =: I_1^\varepsilon + I_2^\varepsilon,$$

where $\varepsilon > 0$ and we write Δ_x to emphasize that in $\Delta_x f(x-y)$ the Laplacian is with respect to the x variable.

Noticing that $\Delta_x f(x-y) = \Delta_y f(x-y)$, Green's identities give:

$$I_1 = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y)\, \Delta_y f(x-y)\, dy = -\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \nabla \Gamma(y) \cdot \nabla_y f(x-y)\, dy$$

$$+ \int_{\partial B_\varepsilon(0)} \Gamma(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y) =: I_{11}^\varepsilon + I_{12}^\varepsilon,$$

where we write ∇_y and $dS(y)$ to emphasize that the gradient and integration over $\partial B_\varepsilon(0)$ are on the y variable. We also notice that in the integration by parts there is no term to be "evaluated at ∞" since f has compact support.

Let's now analyze the integrals I_2^ε, I_{11}^ε, and I_{12}^ε. Observe that:

$$|I_2^\varepsilon| \leq \int_{B_\varepsilon(0)} |\Gamma(y)| \overbrace{|\Delta_x f(x-y)|}^{\leq C'} dy \leq C' \int_{B_\varepsilon(0)} |\Gamma(y)|\, dy$$

$$\leq C' \int_0^\varepsilon \frac{1}{r^{n-2}} r^{n-1}\, dr = C' \varepsilon^2.$$

Since $dS(y) = \varepsilon^{n-1} d\omega$ and $|\Gamma(y)| \leq C'/\varepsilon^{n-2}$ on $\partial B_\varepsilon(0)$:

$$|I_{12}^\varepsilon| \leq \int_{\partial B_\varepsilon(0)} |\Gamma(y)| \left| \frac{\partial f}{\partial \nu}(x-y) \right| dS(y) \leq C' \varepsilon.$$

For I_{11}^{ε}, we integrate by parts again:

$$I_{11}^{\varepsilon} = -\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\,dy = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta\Gamma(y)\, f(x-y)\,dy$$

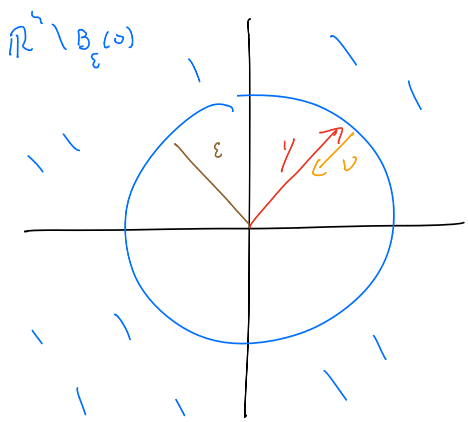
$$-\int_{\partial B_\varepsilon(0)} \frac{\partial\Gamma}{\partial\nu}(y)\, f(x-y)\,dS(y) = 0 - \int_{\partial B_\varepsilon(0)} \frac{\partial\Gamma(y)}{\partial\nu} f(x-y)\,dS(y),$$

where we used that $\Delta\Gamma(y) = 0$ for $y \neq 0$.

From the explicit expression for $\Gamma(y)$, compute:

$$\nabla\Gamma(y) = \frac{1}{n\omega_n}\frac{y}{|y|^n}, \quad y \neq 0.$$

The unit outer normal in the integral is given by $\nu = \frac{-y}{|y|}$, thus



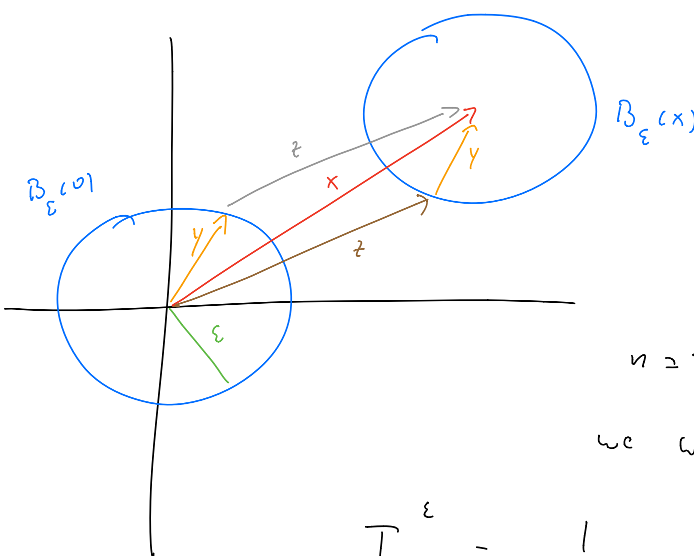
$$I_{11}^{\varepsilon} = \int_{\partial B_\varepsilon(0)} \frac{1}{n\omega_n}\frac{|y|^2}{|y|^{n+1}} f(x-y)\,dS(y)$$

$$= \frac{1}{n\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)\,dS(y),$$

since $|y| = \varepsilon$ on $\partial B_\varepsilon(0)$.

Making a change of variables $x-y=z$, we find

$$I_{11}^{\varepsilon} = \frac{1}{n\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} f(z)\, dS(z)$$



Note that $n\omega_n \varepsilon^{n-1}$ is the surface are, or volume, of $\partial B_\varepsilon(x)$ (e.g., for $n=3$, $n\omega_n\varepsilon^{n-1} = 4\pi\varepsilon^2$), so we write

$$I_{11}^{\varepsilon} = \frac{1}{\mathrm{vol}(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} f(y)\, dS(y).$$

Since we have

$$\Delta u(x) = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} (\cdots) + \int_{B_\varepsilon(0)} (\cdots) = I_1^\varepsilon + I_2^\varepsilon$$

which is valid for any $\varepsilon > 0$, we conclude that

$$\Delta u(x) = \lim_{\varepsilon \to 0^+} I_1^\varepsilon + \lim_{\varepsilon\to 0^+} I_2^\varepsilon$$

if the limits exist. From the foregoing:

$$\lim_{\varepsilon\to 0^+} I_2^\varepsilon = 0,$$

$$\lim_{\varepsilon\to 0^+} I_1^\varepsilon = \lim_{\varepsilon\to 0^+} I_{11}^\varepsilon + \underbrace{\lim_{\varepsilon\to 0^+} I_{12}^\varepsilon}_{=0}$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{vol(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} f(y)\, dS(y).$$

The result (iii) now follows from the Lemma stated right below whose proof will be a HW. $\square$

<u>Lemma.</u> For any continuous function h:

$$\lim_{\varepsilon\to 0^+} \frac{1}{vol(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} h(y)\, dS(y) = h(x),$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{vol(B_\varepsilon(x))} \int_{B_\varepsilon(x)} h(y)\, dy = h(x).$$

<u>proof:</u> HW.

<u>Remark.</u> From the expression for $\Gamma(x)$ we obtain the following useful estimates:

$$|D\Gamma(x)| \le \frac{C'}{|x|^{n-1}}, \quad |D^2\Gamma(x)| \le \frac{C'}{|x|^n}, \quad x \ne 0.$$

Harmonic functions

Def. A solution to Laplace's equation is called a harmonic function. We say that u is a harmonic function (or simply that u is harmonic) in Ω if we want to emphasize that it solves Laplace's equation in Ω.

Theo (mean value formula for Laplace's equation). Let $u \in C^2(\Omega)$ be harmonic in Ω. Then

$$u(x) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r} u \, dS = \frac{1}{\operatorname{vol}(B_r(x))} \int_{B_r(x)} u \, dy,$$

for each $\overline{B_r(x)} \subset \Omega$.

Remark. This theorem says that harmonic functions are "non-local" since their value at x depends on their values on $\partial B_r(x)$; in particular r can be arbitrarily large for $\Omega = \mathbb{R}^n$.

Proof. Define

$$f(r) := \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dS(y)$$

Changing variables $z = \frac{y-x}{r}$, recalling that

$dS = r^{n-1}\, d\omega$, $\mathrm{vol}(\partial B_r(x)) = n\omega_n r^{n-1}$:

$$f(r) = \frac{1}{n\omega_n} \int_{\partial B_1(0)} u(x+rz)\, dS(z).$$

Taking the derivative and noticing that we can differentiate under the integral:

$$f'(r) = \frac{1}{n\omega_n} \int_{\partial B_1(0)} \nabla u(x+rz)\cdot z\, dS(z).$$

Changing variables back to y:

$$f'(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} \nabla u(y)\cdot \left(\frac{y-x}{r}\right) dS(y).$$

Since $\frac{y-x}{r} = \nu =$ unit outer normal to $\partial B_r(x)$:

$$f'(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} \nabla u(y) \cdot \nu \, dS(y)$$

$$= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}(y) \, dS(y)$$

$$= \frac{1}{n\omega_n r^{n-1}} \int_{B_r(x)} \Delta u(y) \, dy = 0$$

where we used Green's identities. Thus $f(r)$ is constant so

$$\frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \, dS = f(r) = \lim_{r \to 0^+} f(r) = \lim_{r \to 0^+} \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \, dS$$

$$= u(x),$$

showing the first equality. For the second, integrate in polar coordinates to find

$$\frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} u(y) \, dy = \frac{1}{\omega_n r^n} \int_0^r \Big(\underbrace{\int_{\partial B_s(x)} u \, dS}_{= n\omega_n s^{n-1} u(x)} \Big) \, ds = u(x).$$

$\square$

Theo. (converse of the mean value property). If $u \in C^2(\Omega)$ is such that $u(x) = \dfrac{1}{\mathrm{vol}(\partial B_r(x))} \displaystyle\int_{\partial B_r(x)} u \, dS$

for each $\overline{B_r(x)} \subset \Omega$, then u is harmonic.

proof. This will be a HW.

Def. Let $\mathcal{U} \subseteq \mathbb{R}^n$. We say that a subset $V \subseteq \mathcal{U}$ is relatively open, or open in $\mathcal{U}$, if $V = \mathcal{U} \cap W$ for some open set $W \subseteq \mathbb{R}^n$. $V \subseteq \mathcal{U}$ is said to be relatively closed, or closed in $\mathcal{U}$, if $V = \mathcal{U} \cap W$ for some closed set $W \subseteq \mathbb{R}^n$. A set $\Omega \subseteq \mathbb{R}^n$ is called connected if the only non-empty subset of Ω that is both open and closed in Ω is Ω itself.

Remark. Sometimes we say simply that $V \subseteq \mathcal{U}$ is open/closed to mean that it is open/closed in $\mathcal{U}$, i.e., $\mathcal{U}$ is implicitly understood.

Students who have not seen the definition of connected sets are encouraged to think about how the above definition corresponds to the intuition that Ω cannot be "split into separate pieces."

Thm (maximum principle). Suppose that

$$u \in C^2(\Omega) \cap C^0(\bar{\Omega})$$

is harmonic, where Ω is bounded. Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Moreover, if $u(x_0) = \max_{\bar{\Omega}} u$ for some $x_0 \in \Omega$, then u is constant.

Remark. Replacing u by $-u$ we obtain similar statements with min. Thus, we can summarize the maximum principle by saying that a harmonic function achieves its maximum and minimum on the boundary.

proof. Suppose that for some $x_0 \in \Omega$ we have $u(x_0) = M = \max_{\bar{\Omega}} u$. For $0 < r < \operatorname{dist}(x_0, \partial\Omega)$, the mean value property gives:

$$M = u(x_0) = \frac{1}{\operatorname{vol}(B_r(x_0))} \int_{B_r(x_0)} u \, dy \leq M.$$

Equality in $\leq$ happens only if $u(y) = M$ for all $y \in B_r(x_0)$. Therefore the set $A := \{ x \in \Omega \mid u(x) = M \}$ is both open and closed in Ω, thus $A = \Omega$, showing the second statement. The first statement follows from the second.

$\square$

Further results for harmonic functions and Poisson's equation

Here we list a few important results concerning $\Delta u = f$ that we will not prove.

Theo (Liouville's theorem). Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded (i.e., there exists a constant $M \geq 0$ such that $|u(x)| \leq M$ for all $x \in \mathbb{R}^n$). Then u is constant.

Def. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given. The following boundary-value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is called the (inhomogeneous) Dirichlet problem for the Laplacian.

Thm. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a C^3 boundary. Let $f \in C^1(\bar{\Omega})$ and $g \in C^3(\bar{\Omega})$. Then, there exists a unique solution $u \in C^2(\bar{\Omega})$ to the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Remark. To solve Poisson's equation in $\mathbb{R}^n$ we introduced the fundamental solution. One approach to solve the Dirichlet problem is to introduce an analogue of the fundamental solution which takes the boundary into account, known as the Green function.

The wave equation in $\mathbb{R}^n$

Here we will study the Cauchy problem for the wave equation in $\mathbb{R}^n$, i.e.,

$$\begin{cases} \Box u = 0 & \text{in } [0,\infty) \times \mathbb{R}^n, \\ u = u_0 & \text{on } \{t=0\} \times \mathbb{R}^n, \\ \partial_t u = u_1 & \text{on } \{t=0\} \times \mathbb{R}^n, \end{cases}$$

where $\Box := -\partial_t^2 + \Delta$ is called the D'Alembertian (or the wave operator) and $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}$ are given.

The initial conditions can also be stated as

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n.$$

Def. The sets

$$\mathcal{C}_{t_0,x_0} := \{ (t,x) \in (-\infty,+\infty) \times \mathbb{R}^n \mid |x-x_0| \leq |t-t_0| \},$$

$$\mathcal{C}^+_{t_0,x_0} := \{ (t,x) \in (-\infty,+\infty) \times \mathbb{R}^n \mid |x-x_0| \leq t-t_0 \},$$

$$\mathcal{C}^-_{t_0,x_0} := \{ (t,x) \in (-\infty,+\infty) \times \mathbb{R}^n \mid |x-x_0| \leq t_0-t \},$$

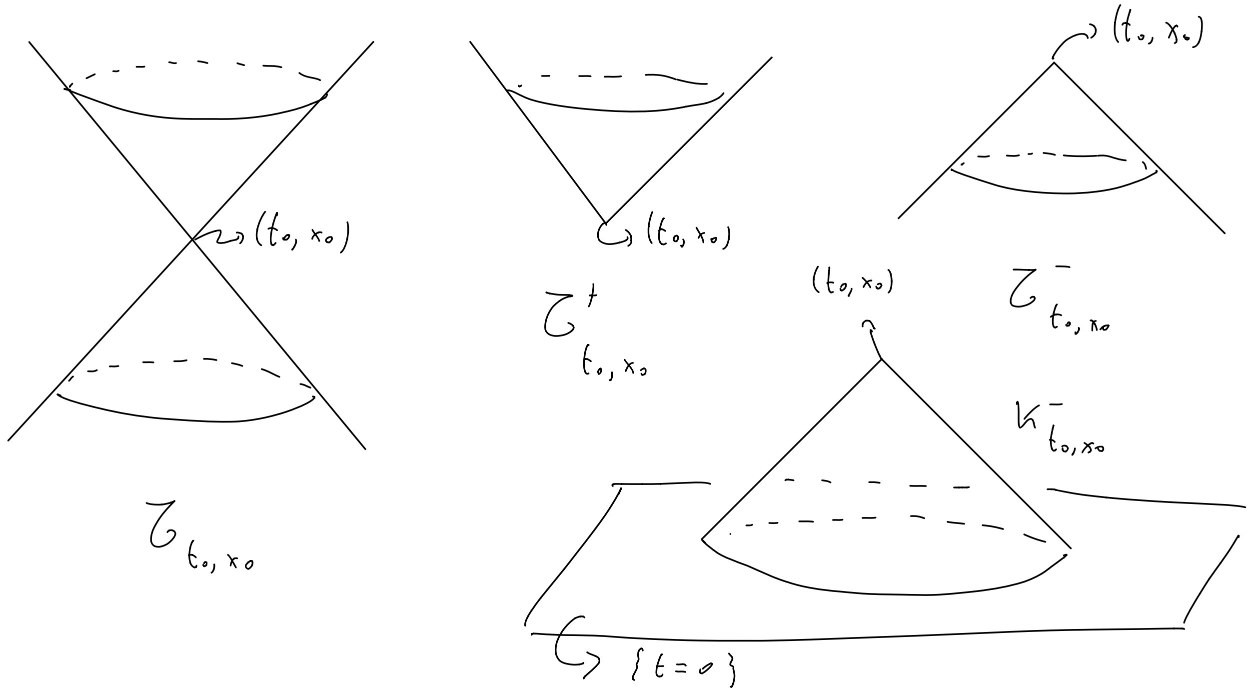
are called, respectively, the light-cone, future light-cone, and <u>past light-cone</u> <u>with vertex at (t_0, x_0)</u>. The sets

$$K_{t_0,x_0} := \mathcal{C}_{t_0,x_0} \cap \{t \geq 0\},$$

$$K^+_{t_0,x_0} := \mathcal{C}^+_{t_0,x_0} \cap \{t \geq 0\},$$

$$K^-_{t_0,x_0} := \mathcal{C}_{t_0,x_0} \cap \{t \geq 0\},$$

are called, respectively, the <u>light-cone</u>, <u>future light-cone</u>, and <u>past light-cone</u> <u>for positive time with vertex at (t_0, x_0)</u>. We often omit "for positive time" and refer to the sets K as light-cones. We also refer to a part of a cone, e.g., for $0 \leq t \leq T$, as the <u>truncated</u> (future, past) light-cone.



Lemma (differentiation of moving regions). Let $\Omega(\tau) \subseteq \mathbb{R}^n$ be a family of bounded domains with smooth boundary depending smoothly on the parameter τ. Let v be the velocity of the moving boundary $\partial\Omega(\tau)$ and ν the unit outer normal to $\partial\Omega(\tau)$. If $f = f(\tau, x)$ is smooth then

$$\frac{d}{d\tau} \int_{\Omega(\tau)} f \, dx = \int_{\Omega(\tau)} \partial_\tau f \, dx + \int_{\partial\Omega(\tau)} f \, v \cdot \nu \, dS.$$

proof: HW. (Compare with the fundamental theorem of calculus).

Theo (finite propagation speed). Let $u \in C^2([0,\infty) \times \mathbb{R}^n)$ be a solution to the Cauchy problem for the wave equation. If $u_0 = u_1 = 0$ on $\{t = 0\} \times B_{t_0}(x_0)$, then $u = 0$ within $K^-_{t_0, x_0}$.
(Thus, the solution at (t_0, x_0) depends only on the data on $B_{t_0}(x_0)$ and the cone $K^-_{t_0,x_0}$ is also called a domain of dependence).

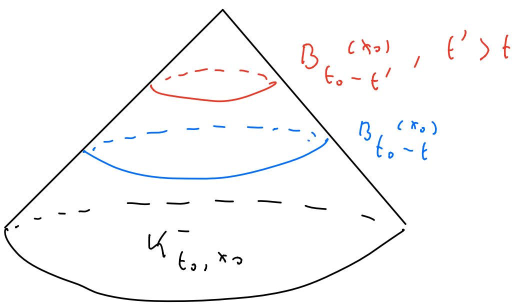
proof: Define the "energy"

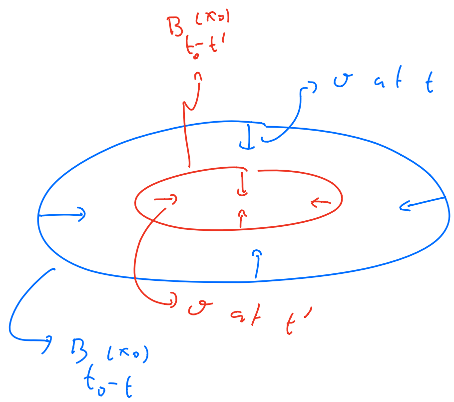
$$E(t) = \frac{1}{2} \int_{B_{t_0 - t}(x_0)} \left((\partial_t u)^2 + |\nabla u|^2 \right) dx, \quad 0 \leq t \leq t_0.$$

Then:

$$\frac{dE}{dt} = \int_{B_{t_0-t}(x_0)} \left(\partial_t u\, \partial_t^2 u + \nabla u \cdot \nabla \partial_t u \right) dx + \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} \left((\partial_t u)^2 + |\nabla u|^2 \right) \sigma \cdot \nu \, dS.$$

The points on the boundary move inward orthogonaly to the the spheres $\partial B_{t_0-t}(x_0)$ and with speed linear in t, thus $\sigma = -\nu$.





Integrating by parts:

$$\int_{B_{t_0-t}(x_0)} \nabla u \cdot \nabla \partial_t u \, dx = - \int_{B_{t_0-t}(x_0)} \Delta u\, \partial_t u \, dx + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial \nu} \partial_t u \, dS$$

Thus

$$\frac{dE}{dt} = \int_{B_{t_0-t}(x_0)} \underbrace{\left(\partial_t^2 u - \Delta u \right)}_{=0} \partial_t u \, dS + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial \nu} \partial_t u$$

$$- \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} \left((\partial_t u)^2 + |\nabla u|^2 \right) dS$$

$$= \int_{\partial B_{t_0-t}(x_0)} \left(\frac{\partial u}{\partial \nu} \partial_t u - \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2 \right) dS$$

$$\leq \int_{\partial B_{t_0-t}(x_0)} \left(|\nabla u|\,|\partial_t u| - \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2 \right) dS,$$

where we used that $\frac{\partial u}{\partial \nu}\partial_t u \leq \left|\frac{\partial u}{\partial \nu}\partial_t u\right| = \left|\frac{\partial u}{\partial \nu}\right| |\partial_t u|$ and $\left|\frac{\partial u}{\partial \nu}\right| = |\nabla u \cdot \nu| \leq |\nabla u| \underbrace{|\nu|}_{=1} = |\nabla u|$. Now apply the Cauchy-Schwarz inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = |\nabla u|$, $b = |\partial_t u|$, to get

$$\frac{dE}{dt} \leq \int_{\partial B_{t_0-t}(x_0)} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2 \right) = 0,$$

thus $E(t)$ is decreasing. Since $E(t) \geq 0$ and

$$E(0) = \frac{1}{2}\int_{\partial B_{t_0}(x_0)} \left(\underbrace{(\partial_t u(0,x))^2}_{= u_1(x) = 0} + \underbrace{|\nabla u(0,x)|^2}_{= |\nabla u_0(0,x)| = 0} \right) dx = 0$$

we conclude that $E(t) = 0$ for all $0 \leq t \leq t_0$.

Since $E(t)$ is the integral of a positive continuous function over $B_{t-t_0}(x_0)$, $E(t)=0$ implies that, for each t, the integrand must vanish, i.e.,

$$(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 = 0 \text{ for all } (t,x) \in K^-_{t_0,x_0},$$

which then implies

$$\partial_t u(t,x) = 0 \text{ and } \nabla u(t,x) = 0 \text{ for all } (t,x) \in K^-_{t_0,x_0}.$$

Since $K^-_{t_0,x_0}$ is connected, we conclude that u is constant in time and space within $K^-_{t_0,x_0}$, i.e. $u(t,x) = C_1 = \text{constant}$ in $K^-_{t_0,x_0}$. Since $u(0,x) = u_0(x) = 0$, C_1 must be zero. $\square$

Notation. Henceforth, we assume that $n \geq 2$. Set

$$\mathcal{U}(t,x;r) := \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(t,y)\, dS(y),$$

$$\mathcal{U}_0(x;r) := \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u_0(t,y)\, dS(y),$$

$$\mathcal{U}_1(x;r) := \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u_1(t,y)\, dS(y),$$

which are spherical averages over $\partial B_r(x)$.

Prop (Euler-Poisson-Darboux equation). Let $u \in C^m([0,\infty)\times\mathbb{R}^n)$, $m \geq 2$, be a solution to the Cauchy problem for the wave equation. For fixed $x \in \mathbb{R}^n$, consider $U = U(t,x;r)$ as a function of t and r. Then $U \in C^m([0,\infty)\times[0,\infty))$ and U satisfies the Euler-Poisson-Darboux equation:

$$\begin{cases} \partial_t^2 U - \partial_r^2 U - \frac{n-1}{r}\partial_r U = 0 & \text{in } (0,\infty)\times(0,\infty), \\ U = U_0 & \text{on } \{t=0\}\times(0,\infty), \\ \partial_t U = U_1 & \text{on } \{t=0\}\times(0,\infty). \end{cases}$$

proof: Differentiability with respect to t is immediate, as is differentiability w.r.t. r for $r>0$.

Arguing as in the proof of the mean value formula for Laplace's equation:

$$\partial_r U(t,x;r) = \frac{r}{n}\frac{1}{\mathrm{vol}(B_r(x))}\int_{B_r(x)} \Delta u(t,y)\,dy.$$

This implies $\lim_{r\to 0^+} \partial_r U(t,x;r) = 0$. Next,

$$\partial_r^2 U(t,x;r) = \frac{1}{n}\frac{1}{\mathrm{vol}(B_r(x))}\int_{B_r(x)} \Delta u(t,y)\,dy$$

$$+ \frac{r}{n} \partial_r \left(\frac{1}{vol(B_r(x))} \right) \int_{B_r(x)} \Delta u(t,y) + \frac{r}{n} \frac{1}{vol(B_r(x))} \partial_r \int_{B_r(x)} \Delta u(t,y)\, dy.$$

But $\partial_r \int_{B_r(x)} \Delta u(t,y)\, dy = \int_{\partial B_r(x)} \Delta u(t,y)\, dS(y)$, and recall that $vol(B_r(x)) = \omega_n r^n$, so

$$\frac{r}{n} \frac{1}{vol(B_r(x))} = \frac{1}{n \omega_n r^{n-1}} = \frac{1}{vol(\partial B_r(x))},$$

$$\frac{r}{n} \partial_r \left(\frac{1}{vol(B_r(x))} \right) = \frac{r}{n} \partial_r \frac{1}{\omega_n r^n} = -\frac{1}{\omega_n r^n} = -\frac{1}{vol(B_r(x))}, \text{ so}$$

$$\partial_r^2 U(t,x;r) = \left(\frac{1}{n} - 1 \right) \frac{1}{vol(B_r(x))} \int_{B_r(x)} \Delta u(t,y)\, dy$$

$$+ \frac{1}{vol(\partial B_r(x))} \int_{\partial B_r(x)} \Delta u(t,y)\, dS(y).$$

This implies that $\lim_{r \to 0^+} \partial_r^2 U(t,x;r) = \frac{1}{n} \Delta u(t,x)$.

Proceeding this way we compute all derivatives of U w.r.t. r and conclude that $U \in C^m([0,\infty) \times [0,\infty))$.

Returning to the expression for $\partial_r U$:

$$\partial_r U = \frac{r}{n} \frac{1}{vol(B_r(x))} \int_{B_r(x)} \Delta u = \frac{r}{n} \frac{1}{vol(B_r(x))} \int_{B_r(x)} \partial_t^2 u, \text{ then}$$

$$\partial_r \left(r^{n-1} \partial_r U \right) = \partial_r \left(\frac{r^n}{n\, vol(B_r(x))} \int_{B_r(x)} \partial_t^2 u \right) = \partial_r \left(\frac{1}{n \omega_n} \int_{B_r(x)} \partial_t^2 u \right)$$

$$= \frac{1}{n \omega_n} \int_{\partial B_r(x)} \partial_t^2 u = \frac{r^{n-1}}{vol(\partial B_r(x))} \int_{\partial B_r(x)} \partial_t^2 u$$

$$= r^{n-1} \partial_t^2 \left(\frac{1}{vol(\partial B_r(x))} \int_{\partial B_r(x)} u \right) = r^{n-1} \partial_t^2 U.$$

On the other hand:

$$\underset{\displaystyle r^{n-1} \partial_t^2 U}{\underset{\shortparallel}{\partial_r \left(r^{n-1} \partial_r U \right)}} = (n-1) r^{n-2} \partial_r U + r^{n-1} \partial_r^2 U,$$

which gives the result. $\square$

Reflection method

We will use the function $U(t,x;r)$ to reduce the higher dimensional wave equation to the 1d wave equation, for which D'Alembert's formula is available, in the variables t and r. However, $U(t,x;r)$ is defined only for $r \geq 0$, whereas D'Alembert's formula is for $-\infty < r < \infty$. Thus, we first consider:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0,\infty) \times (0,\infty), \\ u = u_0 & \text{on } \{t=0\} \times (0,\infty), \\ \partial_t u = u_1 & \text{on } \{t=0\} \times (0,\infty), \\ u = 0 & \text{on } (0,\infty) \times \{x=0\}, \end{cases}$$

where $u_0(0) = u_1(0) = 0$. Consider odd extensions, where $t \geq 0$:

$$\tilde{u}(t,x) = \begin{cases} u(t,x), & x \geq 0 \\ -u(t,-x), & x \leq 0 \end{cases}, \quad \tilde{u}_0 = \begin{cases} u_0(x), & x \geq 0, \\ -u_0(-x), & x \leq 0, \end{cases} \quad \tilde{u}_1(x) = \begin{cases} u_1(x), & x \geq 0 \\ -u_1(x), & x \leq 0. \end{cases}$$

A solution to the problem on $(0,\infty) \times (0,\infty)$ is obtained by solving

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } (0,\infty) \times \mathbb{R}, \\ \tilde{u} = \tilde{u}_0 & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t \tilde{u} = \tilde{u}_1 & \text{on } \{t=0\} \times \mathbb{R}, \end{cases}$$

and restricting to $(0,\infty) \times (0,\infty)$ where $\tilde{u} = u$.

D'Alembert's formula gives

$$\tilde{u}(t,x) = \frac{1}{2}\left(\tilde{u}_0(x+t) + \tilde{u}_0(x-t) \right) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{u}_1(y)\,dy .$$

Consider now $t \geq 0$ and $x \geq 0$, so that $\tilde{u}(t,x) = u(t,x)$. Then $x+t \geq 0$ so that $\tilde{u}_0(x+t) = u_0(x+t)$. If $x \geq t$, then the variable of integration y satisfies $y \geq 0$, since $y \in [x-t, x+t]$. In this case $\tilde{u}_1(y) = u_1(y)$. Thus

$$u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t) \right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy \quad \text{for } x \geq t.$$

If $0 \leq x \leq t$, then $\tilde{u}_0(x-t) = -\tilde{u}_0(-(x-t))$ and

$$\int_{x-t}^{x+t} \tilde{u}_1(y)\,dy = \int_{x-t}^{0} \tilde{u}_1(y)\,dy + \int_{0}^{x+t} \tilde{u}_1(y)\,dy = -\int_{x-t}^{0} u_1(-y)\,dy + \int_{0}^{x+t} u_1(y)\,dy$$

$$= \int_{-x+t}^{0} u_1(y)\,dy + \int_{0}^{x+t} u_1(y)\,dy = \int_{-x+t}^{x+t} u_1(y)\,dy.$$ Thus

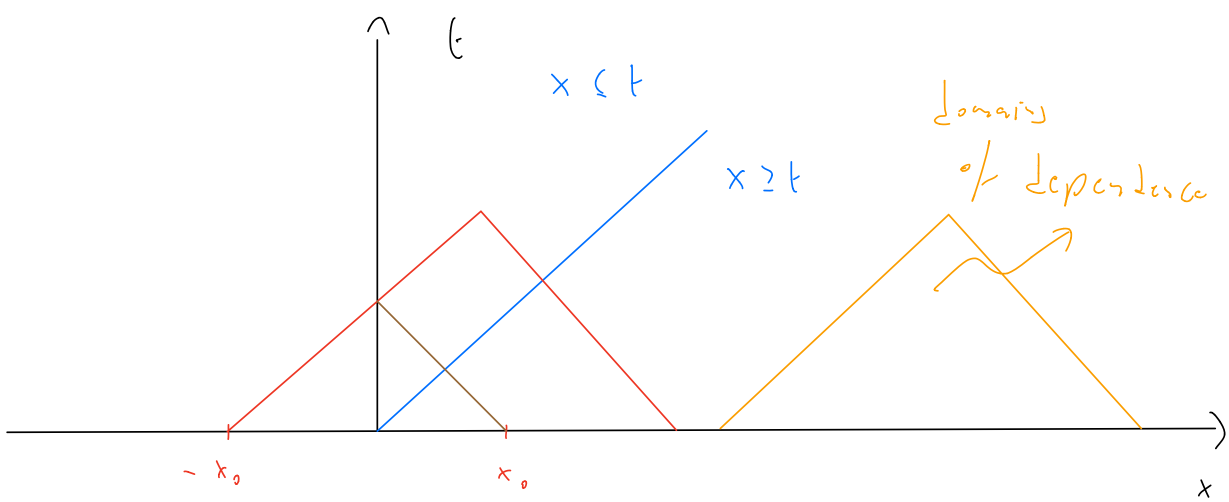
$$u(t,x) = \frac{1}{2}\left(u_0(x+t) - u_0(t-x) \right) + \frac{1}{2}\int_{-x+t}^{x+t} u_1(y)\,dy \quad \text{for } 0 \leq x \leq t.$$

Summarizing:

$$u(t,x) = \begin{cases} \frac{1}{2}\left(u_0(x+t) + u_0(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy, & x \geq t \geq 0, \\ \frac{1}{2}\left(u_0(x+t) - u_0(t-x)\right) + \frac{1}{2}\int_{-x+t}^{x+t} u_1(y)\,dy, & 0 \leq x \leq t. \end{cases}$$

Note that u is not C^2 except if $u_0''(0) = 0$. Note also that $u(t,0) = 0$.

This solution can be understood as follows: for $x \geq t \geq 0$, finite propagation speed implies that the solution "does not see" the boundary. For $0 \leq x \leq t$, the waves traveling to the left are reflected on the boundary where $u = 0$



Solution for $n=3$: Kirchhoff's formula

Set $\tilde{u} = r\mathcal{U}$, $\tilde{u}_0 = r\mathcal{U}_0$, $\tilde{u}_1 = r\mathcal{U}_1$,

where $\tilde{u}, \tilde{u}_0, \tilde{u}_1$ are as above. Then

$$\partial_t^2 \tilde{u} = r\partial_t^2 \mathcal{U} = r\left(\partial_r^2 \mathcal{U} + \frac{3-1}{r}\partial_r \mathcal{U}\right)$$

$$= r\partial_r^2 \mathcal{U} + 2\partial_r \mathcal{U}$$

$$= \partial_r^2 (r\mathcal{U}) = \partial_r^2 \tilde{u},$$

so $\tilde{u}$ solves the 1d wave equation on $(0,\infty)\times(0,\infty)$ with initial conditions $\tilde{u}(0,r) = \tilde{u}_0(r)$, $\partial_t \tilde{u}(0,r) = \tilde{u}_1(r)$.

By the reflection method discussed above, we have

$$\tilde{u}(t,x;r) = \frac{1}{2}\left(\tilde{u}_0(r+t) - \tilde{u}_0(t-r)\right) + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{u}_1(y)\,dy$$

for $0 \le r \le t$, where we used the notation $\tilde{u}_0(r+t)$ and $\tilde{u}_1(y)$ for $\tilde{u}_0(x; r+t)$, $\tilde{u}_1(x;y)$.

From the definition of $\tilde{u}$ and $\mathcal{U}$ and the above formula:

$$u(t,x) = \lim_{r\to 0^+} \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(t,y)\, dS(y)$$

$$= \lim_{r\to 0^+} \mathcal{U}(t,x;r)$$

$$= \lim_{r\to 0^+} \frac{\tilde{\mathcal{U}}(t,x;r)}{r}$$

$$= \lim_{r\to 0^+} \frac{1}{2} \frac{\tilde{\mathcal{U}}_0(t+r) - \tilde{\mathcal{U}}_0(t-r)}{r} + \lim_{r\to 0^+} \frac{1}{2r} \int_{t-r}^{t+r} \tilde{\mathcal{U}}_1(y)\, dy.$$

Note that

$$\lim_{r\to 0^+} \frac{\tilde{\mathcal{U}}_0(t+r) - \tilde{\mathcal{U}}_0(t-r)}{2r} = \lim_{r\to 0^+} \frac{\tilde{\mathcal{U}}_0(t+2r) - \tilde{\mathcal{U}}_0(t)}{2r}$$

$$= \tilde{\mathcal{U}}_0'(t)$$

and

$$\lim_{r\to 0^+} \frac{1}{2r} \int_{t-r}^{t+r} \tilde{\mathcal{U}}_1(y)\, dy = \tilde{\mathcal{U}}_1(t)$$ (this equality is

simply $\lim_{r\to 0^+} \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} f(y)\, dy = f(x)$ for $n=1$). So

$$u(t,x) = \tilde{\mathcal{U}}_0'(t) + \tilde{\mathcal{U}}_1(t).$$

Invoking the definition of $\tilde{u}_0$ and $\tilde{u}_1$:

$$\boxed{u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) \right) + \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_1(y)\, dS(y).}$$

Making the change of variables $z = \frac{y-x}{t}$ (recall that we are treating the $n=3$ case, so in the calculations that follow $n=3$, but we write n for the sake of a cleaner notation):

$$\begin{aligned}
\frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) &= \frac{1}{n\omega_n t^{n-1}} \int_{\partial B_t(x)} u_0(y)\, dS(y) \\
&= \frac{1}{n\omega_n t^{n-1}} \int_{\partial B_1(0)} u_0(x+tz)\, t^{n-1}\, dS(z) \\
&= \frac{1}{n\omega_n} \int_{\partial B_1(0)} u_0(x+tz)\, dS(z).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) \right) &= \frac{1}{n\omega_n} \frac{\partial}{\partial t} \int_{\partial B_1(0)} u_0(x+tz)\, dS(z) \\
&= \frac{1}{n\omega_n} \int_{\partial B_1(0)} \nabla u_0(x+tz)\cdot z\, dS(z).
\end{aligned}$$

Changing variables back to y, i.e., $y = x+tz$ and recalling that $dS(y) = t^{n-1}\, dS(z)$:

$$\frac{\partial}{\partial t}\left(\frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} u_0(y)\, dS(y)\right) = \frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} \nabla u_0(y)\cdot\left(\frac{y-x}{t}\right) dS(y).$$

Using this in the above expression for $u(t,x)$:

$$\boxed{\begin{aligned} u(t,x) &= \frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} \big(u_0(y) + t\, u_1(y)\big)\, dS(y) \\ &\quad + \frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} \nabla u_0(y)\cdot(y-x)\, dS(y), \end{aligned}}$$

which is known as Kirchhoff's formula.

Theo. Let $u_0 \in C^3(\mathbb{R}^3)$ and $u_1 \in C^2(\mathbb{R}^3)$. Then, there exists a unique $u \in C^2([0,\infty)\times\mathbb{R}^3)$ that is a solution to the Cauchy problem for the wave equation in three spatial dimensions. Moreover, u is given by Kirchhoff's formula.

proof: Define u by Kirchhoff's formula. By construction it is a solution with the stated regularity. Uniqueness follows from the finite speed propagation property. $\square$

Solution for $n=2$: Poisson's formula

We now consider $u \in C^2([0,\infty)\times\mathbb{R}^2)$ a solution to the wave equation for $n=2$. Then

$$v(t,x^1,x^2,x^3) := u(t,x^1,x^2)$$

is a solution for the wave equation in $n=3$ dimensions with data $v_0(x^1,x^2,x^3) := u_0(x^1,x^2)$ and $v_1(x^1,x^2,x^3) := u_1(x^1,x^2)$. Let us write $x=(x^1,x^2)$ and $\bar{x}=(x^1,x^2,0)$ Thus, from the $n=3$ case:

$$u(t,x) = v(t,\bar{x}) = \frac{\partial}{\partial t}\left(\frac{t}{\mathrm{vol}(\partial \bar{B}_t(\bar{x}))}\int_{\partial \bar{B}_t(\bar{x})} v_0\, d\bar{S}\right) + \frac{t}{\mathrm{vol}(\partial \bar{B}_t(\bar{x}))}\int_{\partial \bar{B}_t(\bar{x})} v_1\, d\bar{S},$$

where $\bar{B}_t(\bar{x})$ = ball in $\mathbb{R}^3$ with center $\bar{x}$ and radius t, $d\bar{S}$ = volume element on $\partial\bar{B}_t(\bar{x})$. We now rewrite this formula with integrals involving only variables in $\mathbb{R}^2$.

The integral over $\partial\bar{B}_t(\bar{x})$ can be written as

$$\int_{\partial \bar{B}_t(\bar{x})} = \int_{\partial \bar{B}^+_t(\bar{x})} + \int_{\partial \bar{B}^-_t(\bar{x})},$$

where $\partial\bar{B}^+_t(\bar{x})$ and $\partial\bar{B}^-_t(\bar{x})$ are, respectively, the upper and lower hemispheres of $\partial\bar{B}_t(\bar{x})$.

The upper cap $\partial \bar{B}^+_t(\bar{x})$ is parametrized by

$$f(y) = \sqrt{t^2 - |y-x|^2}, \quad y = (y^1, y^2) \in B_t(x), \quad x = (x^1, x^2),$$

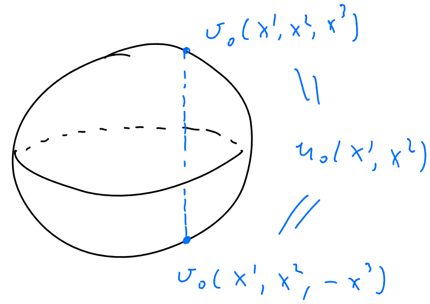
where $B_t(x)$ is the ball of radius t and center x in $\mathbb{R}^2$.

Recalling the formula for integrals along a surface given by a graph:

$$\frac{1}{\operatorname{vol}(\partial \bar{B}_t(\bar{x}))} \int_{\partial \bar{B}^+_t(\bar{x})} v_0 \, d\bar{S} = \frac{1}{4\pi t^2} \int_{B_t(x)} u_0(y) \sqrt{1 + |\nabla f(y)|^2} \, dy,$$

where we used that $v_0(x^1, x^2, x^3) = u_0(x^1, x^2)$. This last fact also implies that

$$\int_{\partial \bar{B}^+_t(\bar{x})} v_0 \, d\bar{S} = \int_{\partial \bar{B}^-_t(\bar{x})} v_0 \, d\bar{S},$$



Thus:

$$\frac{1}{\operatorname{vol}(\partial \bar{B}_t(\bar{x}))} \int_{\partial \bar{B}_t(\bar{x})} v_0 \, d\bar{S} = \frac{2}{4\pi t^2} \int_{B_t(x)} u_0(y) \sqrt{1 + |\nabla f(y)|^2} \, dy$$

$$= \frac{1}{2\pi t} \int_{B_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} \, dy.$$

In the last step we used

$$1 + |\nabla f(y)|^2 = 1 + \frac{|y-x|^2}{t^2 - |y-x|^2} = \frac{t^2}{t^2 - |y-x|^2}.$$

Similarly

$$\frac{t}{\operatorname{vol}(\partial \bar{B}_t(\bar{x}))} \int_{\partial \bar{B}_t(\bar{x})} v_1 \, d\bar{S} = \frac{1}{2\pi} \int_{B_t(x)} \frac{u_1(y)}{\sqrt{t^2 - |y-x|^2}} \, dy.$$

Hence

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int_{B_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} \, dy \right) + \frac{1}{2\pi} \int_{B_t(x)} \frac{u_1(y)}{\sqrt{t^2 - |y-x|^2}} \, dy$$

$$= \frac{1}{2} \frac{\partial}{\partial t}\left(\frac{t^2}{\operatorname{vol}(B_t(x))} \int_{B_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} \, dy \right) + \frac{1}{2} \frac{t^2}{\operatorname{vol}(B_t(x))} \int_{B_t(x)} \frac{u_1(y)}{\sqrt{t^2 - |y-x|^2}} \, dy.$$

Changing variables $\frac{y-x}{t} = z$ in the first integral (so $dy = t^2 dz$)

$$\frac{\partial}{\partial t}\left(\frac{t^2}{\operatorname{vol}(B_t(x))} \int_{B_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}} \, dy \right) = \frac{\partial}{\partial t}\left(\frac{t}{\operatorname{vol}(B_1(0))} \int_{B_1(0)} \frac{u_0(x+tz)}{\sqrt{1 - |z|^2}} \, dz \right)$$

$$= \frac{1}{\operatorname{vol}(B_1(0))} \int_{B_1(0)} \frac{u_0(x+tz)}{\sqrt{1 - |z|^2}} \, dz + \frac{t}{\operatorname{vol}(B_1(0))} \int_{B_1(0)} \frac{\nabla u_0(x+tz) \cdot z}{\sqrt{1 - |z|^2}} \, dz$$

$$= \frac{t}{\mathrm{vol}(B_t(x))} \int_{B_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}}\, dy + \frac{t}{\mathrm{vol}(B_t(x))} \int_{B_t(x)} \frac{\nabla u_0(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\, dy,$$

where in the last step we changed variables back to y. Hence

$$\boxed{\begin{aligned} u(t,x) = {} & \frac{1}{2} \frac{1}{\mathrm{vol}(B_t(x))} \int_{B_t(x)} \left(\frac{t\, u_0(y) + t^2 u_1(y)}{\sqrt{t^2 - |y-x|^2}} \right) dy \\ & + \frac{1}{2} \frac{1}{\mathrm{vol}(B_t(x))} \int_{B_t(x)} \frac{t\, \nabla u_0(y)(y-x)}{\sqrt{t^2 - |y-x|^2}}\, dy, \end{aligned}}$$

which is known as Poisson's formula.

Theo. Let $u_0 \in C^3(\mathbb{R}^2)$ and $u_1 \in C^2(\mathbb{R}^2)$. Then, there exists a unique $u \in C^2([0,\infty)\times\mathbb{R}^2)$ that is a solution to the Cauchy problem for the wave equation in two spatial dimensions. Moreover, u is given by Poisson's formula.

proof: Define u by Poisson's formula. By construction it is a solution with the stated regularity. Uniqueness follows from the finite speed propagation property. $\square$

Solution for arbitrary $n \geq 2$

The above procedure can be generalized for any $n \geq 2$: for n odd, we show that suitably radially averages of u satisfies a 1d wave equation for $r > 0$ and invoke the reflection principle; for n even, we view u as a solution in $n+1$ dimensions, apply the result for n odd, and then reduce back to n dimensions. The final formulas are

n odd

$$u(t,x) = \frac{1}{\beta_n} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(\frac{t^{n-2}}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0 \, dS \right)$$

$$+ \frac{1}{\beta_n} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(\frac{t^{n-2}}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_1 \, dS \right)$$

where

$$\beta_n := 1 \cdot 3 \cdot 5 \cdots (n-2),$$

n even

$$u(t,x) = \frac{1}{\gamma_n}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\frac{t^n}{\mathrm{vol}(B_t(x))}\int_{B_t(x)}\frac{u_0(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right)$$

$$+\frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\frac{t^n}{\mathrm{vol}(B_t(x))}\int_{B_t(x)}\frac{u_1(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right),$$

where

$$\gamma_n := 2\cdot 4\cdots(n-2)n.$$

Remark. The method of using the solution in $n+1$ to obtain a solution in n dimensions for n even is known as method of descent.

Remark. We alredy know that solutions to the wave equation at (t_0,x_0) depend only on the data on $B_{t_0}(x_0)$. For $n\geq 3$ odd, the above shows that the solution depends only on the data on the boundary $\partial B_{t_0}(x_0)$. This fact is known as the strong Huygens' principle.

The inhomogeneous wave equation

We now consider

$$\begin{cases} \Box u = f & \text{in } (0,\infty)\times\mathbb{R}^n, \\ u = u_0 & \text{on } \{t=0\}\times\mathbb{R}^n, \\ \partial_t u = u_1 & \text{on } \{t=0\}\times\mathbb{R}^n \end{cases}$$

where $f:[0,\infty)\to\mathbb{R}^n$, $u_0, u_1:\mathbb{R}^n\to\mathbb{R}$ are given. f is called a source and this is know as the inhomogeneous Cauchy problem for the wave equation. Since we alredy know how to solve the problem when $f=0$, by linearity it suffices to consider

$$\begin{cases} \Box u = f & \text{in } (0,\infty)\times\mathbb{R}^n, \\ u = 0 & \text{on } \{t=0\}\times\mathbb{R}^n, \\ \partial_t u = 0 & \text{on } \{t=0\}\times\mathbb{R}^n. \end{cases}$$

Let $u_s(t,x)$ be the solution of

$$\begin{cases} \Box u_s = 0 & \text{in } (s,\infty)\times\mathbb{R}^n, \\ u_s = 0 & \text{on } \{t=s\}\times\mathbb{R}^n, \\ \partial_t u_s = f & \text{on } \{t=s\}\times\mathbb{R}^n, \end{cases}$$

This problem is simply the Cauchy proplem with data on $t=s$ instead of $t=0$, so the previous solutions apply.

For $t \geq 0$, define:

$$u(t,x) := \int_0^t u_s(t,x)\, ds.$$

Note that $u(0,x) = 0$. We have

$$\partial_t u(t,x) = u_s(t,x)\Big|_{s=t} + \int_0^t \partial_t u_s(t,x)\, ds.$$

Since $u_s(t,x) = 0$ for $t = s$, the first term vanishes, so

$$\partial_t u(t,x) = \int_0^t \partial_t u_s(t,x)\, ds.$$

Thus $\partial_t u(0,x) = 0$. Taking another derivative:

$$\partial_t^2 u(t,x) = \partial_t u_s(t,x)\Big|_{s=t} + \int_0^t \partial_t^2 u_s(t,x)\, ds.$$

Since $\partial_t u_s\Big|_{s=t} = f(s,x) = f(t,x)$ and $\partial_t^2 u_s = \Delta u_s$:

$$\begin{aligned}
\partial_t^2 u(t,x) &= f(t,x) + \int_0^t \Delta u_s(t,x)\, ds \\
&= f(t,x) + \Delta \int_0^t u_s(t,x)\, ds \\
&= f(t,x) + \Delta u(t,x), \quad \text{i.e.,}
\end{aligned}$$

$$\partial_t^2 u - \Delta u = f.$$

Therefore, we conclude that u satisfies the inhomogeneous wave equation with zero initial conditions. We summarize this in the next theorem:

Theo. Let $f \in C^{[\frac{n}{2}]+1}([0,\infty)\times\mathbb{R}^n)$, where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Let u_s be the unique solution to:

$$\begin{cases} \Box u_s = 0 & \text{in } (s,\infty)\times\mathbb{R}^n, \\ u_s = 0 & \text{on } \{t=s\}\times\mathbb{R}^n, \\ \partial_t u_s = f & \text{on } \{t=s\}\times\mathbb{R}^n, \end{cases}$$

and define u by

$$u(t,x) = \int_0^t u_s(t,x)\,ds.$$

Then $u \in C^2([0,\infty)\times\mathbb{R}^n)$ and is a solution to the Cauchy problem for the wave equation with source f and zero initial conditions.

Remark. The procedure of solving the inhomogeneous equation by solving a homogeneous one with initial condition f is known as the Duhamel principle.

Vector fields as differential operators

To proceed further with our study of the wave equation, we need some definitions and tools that we present here.

Consider a vector field $X = (X^1, \dots, X^n)$. Recall that the directional derivative of a function f in the direction of X is

$$\nabla_X f = X \cdot \nabla f = X^i \partial_i f.$$

Note that we have a map that associates to each vector field the corresponding directional derivative, i.e., $X \mapsto \nabla_X$. Observe that this map is linear (e.g., $X + Y \mapsto \nabla_{X+Y} = \nabla_X + \nabla_Y$). Reciprocally, given ∇_X we can extract back the vector field X, $\nabla_X \mapsto X$. We conclude that $X \mapsto \nabla_X$ is a linear isomorphism. Thus, we identify X and ∇_X and think of vector fields as differentiation operators:

$$X = X^i \partial_i = X^i \frac{\partial}{\partial x^i}.$$

In this setting, as for $X = (X^1, \dots, X^n)$, we say that $X = X^i \partial_i$ is C^k if the functions X^i are C^k.

Remark. In differential geometry, where manifolds are conceived abstractly and not as subsets of $\mathbb{R}^3$, vector fields are defined as differential operators.

Def. The composition of vector fields $\mathbb{X}$ and $\mathbb{Y}$, written $\mathbb{X}\mathbb{Y}$, is the differential operator given by

$$(\mathbb{X}\mathbb{Y})(f) := \mathbb{X}(\mathbb{Y}(f)), \text{ i.e.,}$$

$$(\mathbb{X}\mathbb{Y})(f) = \mathbb{X}^i \partial_i (\mathbb{Y}^j \partial_j f),$$

for any C^2 function. We also write $\mathbb{X}\mathbb{Y}f$ for $(\mathbb{X}\mathbb{Y})(f)$.

Remark. Inductively we can consider the composition of an arbitrary number of vector fields, $\mathbb{X}\mathbb{Y}\mathbb{Z}$, etc. Note that in general $\mathbb{X}\mathbb{Y} \neq \mathbb{Y}\mathbb{X}$ and that $\mathbb{X}\mathbb{Y}$ is not a vector field (i.e., in general $\mathbb{X}\mathbb{Y} \neq \nabla_{\mathbb{Z}}$ for some vector field $\mathbb{Z}$).

Prop. Let $\mathbb{X}$ and $\mathbb{Y}$ be C^k vector fields, $k \geq 2$. Then the expression

$$[\mathbb{X}, \mathbb{Y}] := \mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X},$$

called the commutator of $\mathbb{X}$ and $\mathbb{Y}$, is a C^{k-1} vector field.

proof: HW. $\square$

Prop (properties of the commutator). It holds that:

(i). $[X, Y]$ is linear in X and Y.

(ii). $[X, Y] = -[Y, X]$.

(iii) (Jacobi identity)

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

proof: (i) and (ii) are straightforward and (iii) is a direct calculation. $\square$

Def. Let $\mathcal{X} = \{X_1, \dots, X_L\}$ be a collection of smooth vector fields in $\mathbb{R}^n$. Given a non-negative integer $k \geq 0$, define

$$|u(x)|_{\mathcal{X},k} := \sum_{l=1}^{k} \left(\sum_{i_1,\dots,i_l=1}^{L} |X_{i_1} X_{i_2} \cdots X_{i_l} u|^2 \right)^{1/2},$$

for any smooth function $u: \mathbb{R}^n \to \mathbb{R}$. We define the "norm"

$$\|u\|_{\mathcal{X},k} := \left(\int_{\mathbb{R}^n} |u(x)|^2_{\mathcal{X},k} \, dx \right)^{1/2}$$

and write $\|u\|_{\mathcal{X},k} = \infty$ when the integral on the RHS does not converge.

Remark. Above, we wrote "norm" in quotation marks because $\|u\|_{\mathcal{X},k}$ is only a semi-norm. We abuse language and often denote semi-norms by norms. Note that in the particular case $k=1$, $X_i = \partial_i$, $L = n$, we have

$$|u(x)|_{\mathcal{X},k} = \Big(\sum_{i=1}^{n} |\partial_i u(x)|^2 \Big)^{1/2} = |\nabla u(x)|.$$

Remark. Above, we assumed that the X_i's are u are smooth for simplicity. We could consider limited regularity instead. The same is true for much of what follows.

Def and notation. The collection of numbers $g := \{g_{\alpha\beta}\}_{\alpha,\beta=0}^{n}$ where $g_{00} = -1$, $g_{ii} = 1$ $(i = 1,\dots,n)$, and $g_{\alpha\beta} = 0$ otherwise is called the Minkowski metric. It can be identified with the entries of the matrix

$$M = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

The collection $g^{-1} := \{g^{\alpha\beta}\}_{\alpha,\beta=0}^{n}$, where $g^{00} = -1$, $g^{ii} = 1$ $(i=1,\dots,n)$, $g^{\alpha\beta} = 0$ otherwise, which can be identified with the entries of the matrix M^{-1}, is called the inverse Minkowski metric. Given as object with Greek indices (i.e., varying from 0 to n, recall our indice conventions) we can raise and lower indices using g and g^{-1} in analogy with what we did using the Kronecker delta. E.g.:

$$\bar{X}_\alpha := g_{\alpha\beta} \bar{X}^\beta,$$

so that $\bar{X}_0 = -\bar{X}^0$ and $\bar{X}_i = \bar{X}^i$. We define the Minkowski inner product by

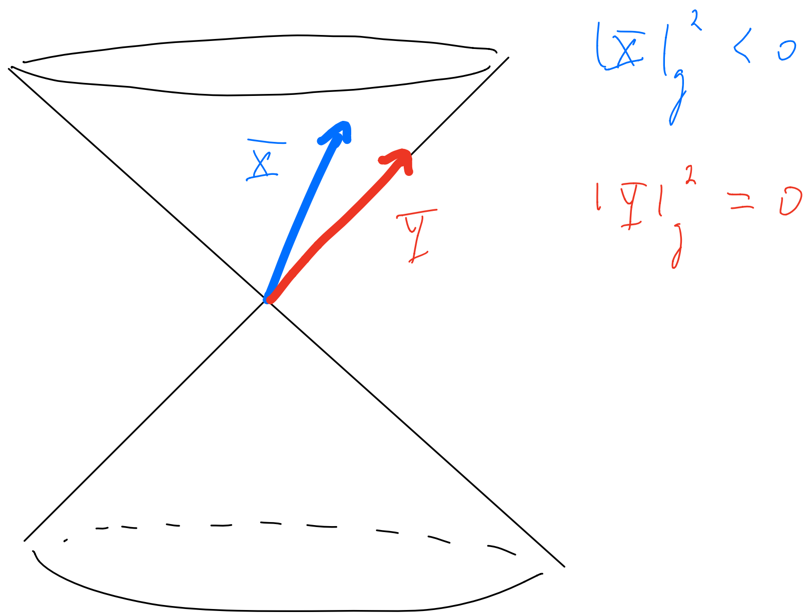
$$\langle \bar{X}, \bar{Y} \rangle_g := g_{\alpha\beta} \bar{X}^\alpha \bar{Y}^\beta = \bar{X}^\alpha \bar{Y}_\alpha$$
$$= -\bar{X}^0 \bar{Y}^0 + \sum_{i=1}^n \bar{X}^i \bar{Y}^i .$$

Note that $\langle \, , \, \rangle_g$ is non-degenerate (like the Euclidean inner product) but it is not positive definite (unlike the Euclidean inner product). We then define the Minkowski norm (squared) as

$$|\bar{X}|_g^2 := \langle \bar{X}, \bar{X} \rangle_g .$$

Vectors such that $|\bar{X}|_g^2 < 0$ are called timelike, $|\bar{X}|_g^2 = 0$ are called null-like, and $|\bar{X}|_g^2 > 0$ spacelike.

Students can check that $\mathcal{C}_{t_0, x_0}$ consists of the set of vectors based at (t_0, x_0) that are timelike or null and $\partial \mathcal{C}_{t_0,x_0}$ consists of the set of vectors based at t_0, x_0 that are null-like.



Remark. The previous identification of vector fields with differential operators and the definitions that follow (commutator, norm, etc.) apply as well for vector fields containing a zeroth component, $\overline{X} = (\overline{X}^0, \overline{X}^1, \dots, \overline{X}^n)$, i.e., vector fields in $\mathbb{R} \times \mathbb{R}^n$ or subsets of it, and functions $u = u(t, x^1, \dots, x^n)$.

The Lorentz vectorfields

We introduce the following vectorfields in $\mathbb{R} \times \mathbb{R}^3$:

- The translations:

$$T_\mu := \frac{\partial}{\partial x^\mu}$$

- The angular momenta:

$$\Omega_{\mu\nu} := x_\mu \partial_\nu - x_\nu \partial_\mu$$

- The dilation

$$S := x^\mu \partial_\mu$$

Among the angular momenta, we distinguish further:

- The spatial rotations, Ω_{ij}:

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i$$

- The boosts or hyperbolic rotations, Ω_{i0}

$$\Omega_{i0} = x_i \partial_t + t \partial_i$$

(the plus comes from $x_0 = g_{0\rho} x^\rho = -x^0 = -t$). Note that

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu}.$$

Together, these vectorfields are called the Lorentz vectorfields (or Lorentz fields). We denote

$$\mathcal{Z} := \{ T_\mu, \Omega_{\mu\nu}, S \}_{\mu,\nu=0}^{n}$$

the set of Lorentz vectorfields.

Notation. Let Ω be an open set in $\mathbb{R}^n$. We denote by $C^\infty(\Omega, \mathbb{R}^m)$ the set of all infinitely many times differentiable (i.e., smooth) maps $u : \Omega \to \mathbb{R}^m$. We put $C^\infty(\Omega) := C^\infty(\Omega, \mathbb{R})$ (although we can abuse notation and write $C^\infty(\Omega)$ for $C^\infty(\Omega, \mathbb{R}^m)$ if $\mathbb{R}^m$ is clear from the context.

Def. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A differential operator $\mathcal{P}$ on Ω is a map $\mathcal{P} : C^\infty(\Omega) \to C^\infty(\Omega)$ of the form

$$(\mathcal{P}u)(x) = P(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x),$$

where $x \in \Omega$, $u \in C^\infty(\Omega)$, and P is a function

$$P : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

The number k above is called the order of the operator. We often identify $\mathcal{P}$ with P and say "the differential operator P."

<u>EX</u>: Take $\Omega = \mathbb{R}^2$. Then

$$Pu = \partial_x^2 u + \partial_y^2 u + u^2$$

is a second-order differential operator. To identify the function P, denote coordinates in $\mathbb{R}^{2^2} \times \mathbb{R}^2 \times \mathbb{R} \times \Omega$ by

$$z = (p_{xx}, p_{xy}, p_{yx}, p_{yy}, p_x, p_y, p, x, y),$$

so $P(z) = p_{xx} + p_{yy} + p^2$.

Observe that the definition of a differential operator takes all entries into account, ignoring, e.g., $\partial_{xy} u = \partial_{yx} u$ etc.

<u>Remarks.</u>

- In the above definition, it is implicitly assumed that the first entry in P is not trivial, so that the order of P is well-defined. Otherwise, we could take, say, the first order operator $Pu = \partial_x u$ and think of it as the second order operator $Pu = 0 \cdot \partial_x^2 u + \partial_x u$, etc.

- P might in fact be defined only on a subset of $C^\infty(\Omega)$, e.g., $Pu = \frac{1}{\partial_x u}$ is not defined on constants. Situations like this will typically be clear from the context.

- We can generalize the above to $C^\infty(\Omega, \mathbb{R}^n)$.

- Differential operators will naturally extend to more general function spaces, e.g., $P : C^k(\Omega) \to C^{k-1}(\Omega)$, whenever the corresponding expressions make sense.

Def. Let P be a differential operator of order k. P is called

- linear, if it has the form

$$(Pu)(x) = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x),$$

for some functions a_α.

- semi-linear, if it has the form

$$(Pu)(x) = \sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0\left(D^{k-1}u(x), \dots, Du(x), u(x), x\right),$$

for some functions a_α.

- quasi-linear if it has the form

$$(Pu)(x) = \sum_{|\alpha| = k} a_\alpha\left(D^{k-1}u(x), \dots, Du(x), u(x), x\right) D^\alpha u(x)$$

$$+ a_0\left(D^{k-1}u(x), \dots, Du(x), u(x), x\right).$$

- P is fully nonlinear if it depends nonlinearly on derivatives of order k.

Remark. A PDE can be equivalently defined as an equation $Pu = f$, where P is a differential operator and f is a given function.

__Def.__ Let P and Q be differential operators defined on a common domain. Their __commutator__ is the differential operator defined by $[P, Q]u := P(Q(u)) - Q(P(u))$.

__Prop.__ The following identities hold:

$$[T_\mu, T_\nu] = 0$$

$$[T_\mu, \Omega_{\alpha\rho}] = g_{\mu\alpha} T_\rho - g_{\mu\rho} T_\alpha$$

$$[T_\mu, S] = T_\mu$$

$$[\Omega_{\mu\nu}, \Omega_{\alpha\rho}] = g_{\mu\alpha} \Omega_{\rho\nu} - g_{\mu\rho} \Omega_{\alpha\nu} + g_{\nu\alpha} \Omega_{\mu\rho} - g_{\nu\rho} \Omega_{\mu\alpha}$$

$$[\Omega_{\mu\nu}, S] = 0$$

$$[\Box, T_\mu] = 0$$

$$[\Box, \Omega_{\mu\nu}] = 0$$

$$[\Box, S] = 2\Box$$

__proof:__ tedious (but straightforward) calculations. $\Box$

__Remark.__ It follows that if u solves $\Box u = 0$, then $\tilde{u} = Zu$, $Z \in \mathcal{Z}$, also solves the equation, $\Box \tilde{u} = 0$. Because of this, the Lorentz fields are referred to as symmetries of the wave equation.

Decay estimates for the wave equation

We are going to use the Lorentz fields to prove the following.

Theo. Let $k \geq [\frac{n}{2}]+2$ be an integer and let u be smooth solution to the wave equation:

$$\Box u = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^n,$$

$n \geq 2$. Then, there exists a constant C_1, depending only on n, such that

$$|\nabla u(t,x)| \leq C_1 (1+t)^{-\frac{n-1}{2}} \left(\|\nabla u(0,\cdot)\|_{\mathcal{L},k} + \|\partial_t u(0,\cdot)\|_{\mathcal{L},k} \right),$$

$t > 0$, $x \in \mathbb{R}^n$.

The proof will be given in steps.

Def. The L^2-norm of a function $f: U \to \mathbb{R}$ is

$$\|f\|_{L^2(U)} := \left(\int_U |f(x)|^2 \, dx \right)^{1/2}.$$

We write $\|f\|_{L^2(U)} = \infty$ if the RHS does not converge. Sometimes we write $\|f\|_{L^2}$ if U is implicitly understood.

Prop. (Sobolev inequality) Let $k > \frac{n}{2}$ be an integer. There exists a constant $C' > 0$, depending on n and k, such that

$$|u(x)| \leq C' \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2}, \quad \forall x \in \Omega,$$

for any smooth $u : \Omega \to \mathbb{R}$.

proof. The proof may be assigned as a HW. $\square$

To understand Sobolev's inequality, note that in general we should not expect to be able to bound $|u(x)|$ by one of its integrals. E.g., take $\Omega = (0,1)$, $u(x) = \frac{1}{\sqrt[4]{x}}$. Then

$$\int_0^1 |u(x)|^2 dx = \int_0^1 \frac{dx}{\sqrt{x}} = \frac{\sqrt{x}}{1/2}\Big|_0^1 = 2, \text{ i.e., } \|u\|_{L^2} = \sqrt{2}.$$

Since $\frac{1}{\sqrt[4]{x}} \to \infty$ as $x \to 0^+$, we see that there does not exist a constant $C' > 0$ such that $|u(x)| \leq C\|u\|_{L^2}$ for all x, i.e., we cannot control u pointwise by its integrals (in the L^2-sense). The Sobolev inequality says that for functions with a large number of derivatives being integrable, such control is possible.

Notation. Let us denote by $\mho$ the collection of spatial angular momenta operators, i.e.,

$$\mho = \{ \Omega_{ij} \}_{i,j=1}^{n} .$$

Lemma. Let $k \geq \left[\frac{n-1}{2}\right] + 1$. There exists a constant $C_1 > 0$, depending on n and k, such that

$$|u(x)| \leq C_1 \left(\int_{\partial B_1(0)} |u(y)|^2_{\mho, k} \, dS(y) \right)^{1/2} \quad \forall x \in \partial B_1(0),$$

for all smooth functions $u : \partial B_1(0) \to \mathbb{R}$.

proof. Begin by noticing that the derivatives Ω_{ij} are always tangent to $\partial B_1(0)$, so that it makes sense to consider $\Omega_{ij} u$ for u defined on $\partial B_1(0)$. Indeed, recalling that $\partial_i r = \frac{x_i}{r}$, we have

$$\Omega_{ij} r = (x_i \partial_j - x_j \partial_i) r = \frac{x_i x_j}{r} - \frac{x_j x_i}{r} = 0 .$$

Next, split the integral over $\partial B_1(0)$ as the integral over

$\partial B_1(0)$ as an integral over two hemispheres $\partial B_1^+(0)$ and $\partial B_1^-(0)$. Parametrize the integral over each sphere as an integral over $B_1^{n-1}(0)$ (as we did in the method of descent).

The tangent space to $\partial B_1(0)$ at any point is spanned by $n-1$ linearly independent vector fields. Since there are $\frac{n(n-1)}{2}$ linearly independent Ω_{ij}'s, we conclude that σ spans the tangent space to $\partial B_1(0)$. Hence, each integral over $\partial B_1^+(0)$ and $\partial B_1^-(0)$ contains all derivatives, i.e., $D^\alpha u$. Applying Sobolev's inequality (which is allowed since $k \geq \left[\frac{n-1}{2}\right]+1$), we obtain the result. $\square$

<u>Lemma.</u> There exists a constant $C' > 0$, depending on n, such that, for every smooth $u : \mathbb{R}^n \to \mathbb{R}$ and every $x \neq 0$,

$$|u(x)| \leq C' |x|^{-\frac{n-1}{2}} \|u\|^{1/2}_{\sigma, \left[\frac{n-1}{2}\right]+1} \|\nabla u\|^{1/2}_{\sigma, \left[\frac{n-1}{2}\right]+1}$$

<u>proof.</u> Fix $x \in \mathbb{R}^n$. We can write $x = r\omega$, $\omega \in \partial B_1(0)$. For fixed ω,

$$|u(r\omega)|^2 \leq C_1 r^{1-n} \left(\int_0^\infty |u(r'\omega)|^2 (r')^{n-1} dr' \right)^{1/2} \left(\int_0^\infty |\partial_{r'} u(r'\omega)|^2 (r')^{n-1} dr' \right)^{1/2}$$

(see below).

From the previous lemma,

$$|u(r'\omega)|^2 \leq C_1 \int_{\partial B_1(0)} |u(r'\zeta)|^2_{\sigma, [\frac{n-1}{2}]+1} \, dS(\zeta).$$

Moreover,

$$\int_0^\infty (r')^{n-1} dr' \int_{\partial B_1(0)} |u(r'\zeta)|^2_{\sigma, [\frac{n-1}{2}]+1} \, dS(\zeta) = \int_{\mathbb{R}^n} |u(x)|^2_{\sigma, [\frac{n-1}{2}]+1} \, dx.$$

A similar inequality holds for $\partial_{r'} u(r'\omega)$, which implies the result.

It remains to prove the inequality for $|u(r\omega)|^2$. Keeping ω fixed and considering $u(r\omega)$ as a function of r, and noting that we can assume $u(r\omega) \to 0$ as $r \to \infty$ (consistent with finiteness of the integrals of u), we have

$$|u(r\omega)|^2 \leq \left| \int_r^\infty \partial_{r'} (u(r'\omega))^2 \, dr' \right|$$

$$\leq 2 \int_r^\infty |u(r'\omega)|\,|\partial_{r'} u(r'\omega)| \left(\frac{r'}{r}\right)^{n-1} dr'$$

$$\leq \frac{2}{r^{n-1}} \int_r^\infty |u(r'\omega)|\,|\partial_{r'} u(r'\omega)|\,(r')^{n-1}\,dr',$$

where we used that $\frac{r'}{r} \geq 1$ for $r' \geq r$. Let:

$$A = \left(\int_r^\infty |u(r',\omega)|^2 (r')^{n-1}\,dr'\right)^{1/2}, \quad B = \left(\int_r^\infty |\partial_{r'} u(r'\omega)|^2 (r')^{n-1}\,dr'\right)^{1/2}.$$

Then:

$$\frac{|u(r'\omega)|}{A}\,\frac{|\partial_{r'} u(r'\omega)|}{B}\,(r')^{n-1} \leq \frac{|u(r'\omega)|^2}{2A^2}(r')^{n-1} + \frac{|\partial_{r'} u(r'\omega)|^2}{2B^2}(r')^{n-1}$$

Integrating w.r.t. r' from r to ∞, the RHS becomes $\frac{1}{2} + \frac{1}{2} = 1$, which implies the desired inequality. $\square$

We now state another type of Sobolev inequality:

Prop (local Sobolev inequality). Let $k > \frac{n}{2}$ be an integer. There exists a constant $C_1 > 0$, depending on n and k, such that for every smooth $u : B_R(0) \to \mathbb{R}$ and all $x \in B_R(0)$:

$$|u(x)| \le C_1 \sum_{i=0}^{k} R^{i - \frac{n}{2}} \left(\int_{B_R(0)} \sum_{|\alpha| = i} |D^\alpha u(x)|^2 \, dx \right)^{1/2}.$$

We will omit the proof of this proposition. The idea is to rescale u to reduce the problem to $B_1(0)$ (this gives the powers of R). We next extend u from $B_1(0)$ to $\mathbb{R}^n$ and show that this extension has norm comparable to that of u in $B_1(0)$.

Lemma. Let $k \ge 0$ be an integer. There exists a constant $C_1 > 0$, depending on n and k, such that for any smooth $u = u(t, \dots, x^n)$, we have

$$|D^\alpha u(t,x)| \le \frac{C_1}{(1 + |r-t|^2)^{\frac{k}{2}}} |u(t,x)|_{\mathcal{Z}, k}$$

for any α such that $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n = k$

__proof.__ By induction in $k \geq 1$. For any (t,x) not on the boundary of the lightcone, we readily check that

$$T_\nu = \frac{1}{r^2 - t^2} \left(x^\mu \Omega_{\mu\nu} + x_\nu S \right).$$

Indeed $x^\mu \Omega_{\mu\nu} + x_\nu S = x^\mu x_\mu \partial_\nu - x^\mu x_\nu \partial_\mu + x_\nu x^\mu \partial_\mu = (r^2 - t^2)\partial_\nu$.

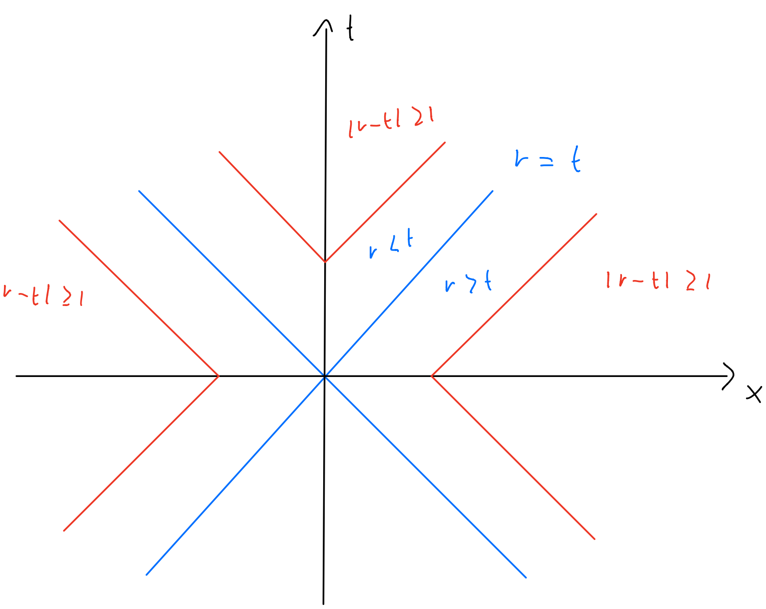
Thus

$$\frac{\partial u}{\partial x^\nu} = \frac{1}{r-t} \frac{1}{r+t} \left(x^\mu \Omega_{\mu\nu} u + x_\nu S u \right).$$

This implies

$$\left| \frac{\partial u}{\partial x^\nu} \right| \leq \frac{C_1}{|r-t|} |u|_{\{\Omega_{\mu\nu}, S\}, 1} \quad \text{for } |r-t| \geq 1.$$

Indeed, observe that $\dfrac{|x^\mu| + |x_\nu|}{r+t} \leq C_1'$ since $|x| = r$.



Since $\dfrac{(1+|r-t|^2)^{1/2}}{|r-t|}$ is a bounded function for

$|r-t| \geq 1$, we obtain the inequality for $|r-t| \geq 1$.

For $|r-t| < 1$, it holds that $\left|\dfrac{\partial u}{\partial x^\nu}\right| \leq \dfrac{c_1}{(1+|r-t|^2)^{1/2}} \left|\dfrac{\partial u}{\partial x^\nu}\right|$, thus combining both regions

$$\left|\frac{\partial u}{\partial x^\nu}\right| \leq \frac{c_1}{(1+|r-t|^2)^{1/2}} \left|\frac{\partial u}{\partial x^\nu}\right|_{\mathcal{L},1}$$

proving the case $k=1$.

Consider now second derivatives. Applying the case $k=1$ to $T_\mu u = \partial_\mu u$ gives

$$\left|\frac{\partial^2 u}{\partial x^\mu \partial x^\nu}\right| \leq \frac{c_1}{(1+|r-t|^2)^{1/2}} \left|T_\mu u\right|_{\mathcal{L},1}.$$

The RHS involves expressions of the form $\bar{X} T_\mu u$ with $\bar{X} \in \mathcal{L}$. From the commutations relations, a term of the type $\bar{X} T_\mu u$ can be written as

$$\begin{aligned} \bar{X} T_\mu u &= T_\mu \bar{X} u + [\bar{X}, T_\mu] u \\ &= T_\mu \bar{X} u + T u \end{aligned}$$

for some translation T and up to numerical factors in the second term. Applying the case $k=1$ to $\bar{X} u$ gives,

$$|T_p \bar{X} u| \leq \frac{C_1}{(1+|r-t|^2)^{1/2}} |\bar{X} u|_{\mathcal{Z},1} \leq \frac{C_1}{(1+|r-t|^2)^{1/2}} |u|_{\mathcal{Z},2},$$

and we also have

$$|T u| \leq \frac{C_1}{(1+|r-t|^2)^{1/2}} |u|_{\mathcal{Z},1} \leq \frac{C_1}{(1+|r-t|^2)^{1/2}} |u|_{\mathcal{Z},2}.$$

Using the foregoing, we obtain the inequality for $k=2$. We continue this way: to estimate a $k+1$ derivative, we write $D^{k+1} u = T D^k u$, apply the k^{th} case, and use the commutation relations. These commutation relation always give a term of the form $T(\dots)$, for which we can apply the $k=1$ case to get an extra term $(1+|r-t|^2)^{-1/2}$, giving the result. $\square$

Prop. Let $k \geq \left[\frac{n}{2}\right]+1$ be an integer. There exists a constant $C > 0$, depending on n and k, such that for any (t,x) with $t \geq 2|x|$ and any smooth $u : \mathbb{R}^n \to \mathbb{R}$,

$$|u(t,x)| \leq C_1 t^{-\frac{n}{2}} \|u(t,\cdot)\|_{\mathcal{Z},k}.$$

Proof. Let $R = \frac{t}{2}$ and apply the local Sobolev inequality to obtain:

$$|u(t,x)| \leq \sum_{i=0}^{k} R^{i-\frac{n}{2}} \left(\int_{B_R(0)} \sum_{|\alpha|\leq i} |D^\alpha u(t,z)|^2 \, dz \right)^{1/2}$$

From the previous lemma,

$$|D^\alpha u(t,x)| \leq \frac{C_1}{(1+|r-t|^2)^{\frac{i}{2}}} |u(t,x)|_{\mathcal{Z},k} , \quad |\alpha| = i,$$

so that

$$|u(t,x)| \leq C_1 \sum_{i=0}^{k} R^{i-\frac{n}{2}} \left(\int_{B_R(0)} \frac{1}{(1+|r-t|^2)^i} |u(t,z)|^2_{\mathcal{Z},i} \, dz \right)^{1/2}$$

For $|x| \leq \frac{t}{2}$, we have

$$(1+|r-t|^2) \geq (1+|r-t|^2)^{1/2} \geq \frac{t}{2} = R$$

since the least $(1+|r-t|^2)^{1/2}$ can be is when $r = |x| = t/2$ so that $\left(1 + \frac{t^2}{4}\right)^{1/2} \geq \frac{t}{2}$. Thus

$$|u(t,x)| \leq C_1 \sum_{i=0}^{k} R^{i-\frac{n}{2}-i} \left(\int_{B_R(0)} |u(t,z)|^2_{\mathcal{Z},i} \, dz \right)^{1/2}$$

$$\leq C_1 t^{-\frac{n}{2}} \|u(t,\cdot)\|_{\mathcal{Z},k} .$$

$\square$

Prop. Let $k \geq \left[\frac{n}{2}\right] + 2$ be an integer. There exists a constant $C > 0$, depending on k and n, such that for all $t > 0$, $x \in \mathbb{R}^n$, and any smooth $u : \mathbb{R}^n \to \mathbb{R}$, it holds

$$|u(t,x)| \leq C (1+t)^{-\frac{n-1}{2}} \|u(t,\cdot)\|_{\mathcal{L},k} .$$

proof: For $|x| \geq \frac{t}{2}$, the second lemma of this section gives

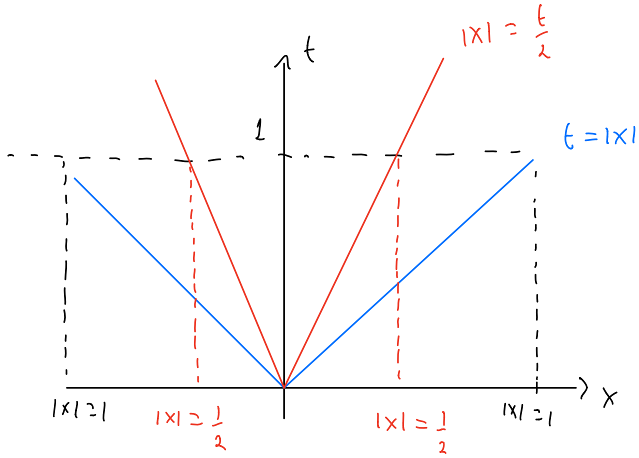
$$|u(t,x)| \leq C |x|^{-\frac{n-1}{2}} \|u\|^{1/2}_{\mathcal{O},\left[\frac{n-1}{2}\right]+1} \|\nabla u\|^{1/2}_{\mathcal{O},\left[\frac{n-1}{2}\right]+1}$$

$$\leq C\, t^{-\frac{n-1}{2}} \|u\|^{1/2}_{\mathcal{L},\left[\frac{n-1}{2}\right]+1} \|\nabla u\|^{1/2}_{\mathcal{L},\left[\frac{n-1}{2}\right]+1}$$

$$\leq C\, t^{-\frac{n-1}{2}} \|u(t,\cdot)\|_{\mathcal{L},k} .$$

For $t \geq 1$, we can replace $t^{-\frac{n-1}{2}}$ by $(1+t)^{-\frac{n-1}{2}}$ in the above inequality, and $t^{-\frac{n}{2}}$ by $(1+t)^{-\frac{n-1}{2}}$ in the inequality of the previous proposition, which was valid for $|x| \leq \frac{t}{2}$.

For $t < 1$, if $|x| \leq \frac{1}{2}$ we can apply Sobolev's inequality on $B_{\frac{1}{2}}(0)$. Finally, for $t < 1$ and $|x| \geq \frac{1}{2}$, so that



$|x| \geq \frac{1}{4}(t+1)$ (since $\frac{t+1}{4} \leq \frac{1}{2}$ for $t < 1$), we apply again the second lemma of this section so that

$$|u(t,x)| \leq C' |x|^{-\frac{n-1}{2}} \|u\|_{\mathcal{Z},k}$$
$$\leq C'(1+t)^{-\frac{n-1}{2}} \|u\|_{\mathcal{Z},k},$$

finishing the proof. $\square$

So far we proved several inequalities valid for an arbitrary u. We will now use a solution of the wave equation to obtain the main result.

<u>Proof of the decay estimate:</u> By the commutation relations between $\mathcal{Z}$ and $\square$, we have that for any collection $\{\mathcal{X}_i\}_{i=1}^{\ell} \subset \mathcal{Z}$,

$$v := \overline{X}_1 \cdots \overline{X}_\ell u$$

satisfies $\Box v = 0$ if $\Box u = 0$. Using conservation of energy for v gives the result. $\Box$

The canonical form of second order linear PDEs and remarks on tools for their study

Consider the linear PDE

$$a^{\mu\nu}\partial_\mu\partial_\nu u + b^\mu\partial_\mu u + cu = f \quad \text{in } \Omega,$$

for $u = u(t,x)$, where the coefficients $a^{\mu\nu}$, b^μ, c, and the source term are given functions of (t,x). We can assume that the coefficients $a^{\mu\nu}$ are symmetric, i.e., $a^{\mu\nu} = a^{\nu\mu}$. (If not, we can define $\tilde{a}^{\mu\nu} = a^{\mu\nu} + a^{\nu\mu}$ and write the PDE with $\tilde{a}^{\mu\nu}$.)

The PDE is called elliptic if it has the form

$$a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu = f$$

and there exists a constant $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. Note that in this case there is no differentiation with respect to t so we can assume all functions to depend only on x.

The PDE is called parabolic if it has the form

$$\partial_t u - a^{ij}\partial_i\partial_j u + b^i \partial_i u + cu = f$$

and there exists a constant $\lambda > 0$ such that

$$a^{ij}(t,x)\,\xi_i \xi_j \geq \lambda |\xi|^2$$

for all $(t,x) \in \Omega$ and all $\xi \in \mathbb{R}^n$.

The PDE is called hyperbolic if it has the form

$$\partial_t^2 u - a^{ij}\partial_i\partial_j u + b^i \partial_i u + cu = f$$

and there exists a constant $\lambda > 0$ such that

$$a^{ij}(t,x)\,\xi_i \xi_j \geq \lambda |\xi|^2$$

for all $(t,x) \in \Omega$ and all $\xi \in \mathbb{R}^n$.

The Poisson, heat, and wave equations are examples of elliptic, parabolic, and hyperbolic PDEs, respectively. In fact, the condition $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ implies that given a point x_0, it is possible to choose x-coordinates such than, in a small neighborhood of x_0 we have

$$a^{ij} \approx \delta^{ij}.$$

Therefore, elliptic, parabolic, and hyperbolic equations can be viewed (in a neighbourhood of x_0) as approximated by the Poisson, heat, and wave equation, respectively. As we discuss below, we can think of elliptic, parabolic, and hyperbolic equations as generalizations of the Poisson, heat, and wave equation.

Note that these definitions depend on the domain Ω, i.e., a certain PDE might be, say, elliptic in a domain Ω but not in another domain Ω'; or not elliptic in Ω, but elliptic in a subdomain $\Omega' \subset \Omega$.

We have not given the most general definitions, but they will suffice for our discussion. (Some generalizations are trivial. E.g., if in a parabolic PDE we had $a^0 \partial_t u$ instead of $\partial_t u$ and $a^0 \neq 0$, we can simply divide by a^0.)

There exists a fairly general theory of elliptic, parabolic, and hyperbolic equations (note that here we are talking about linear equations; it is possible to define non-linear versions of these equations but then much less is known. Compare to ODEs). The important point to keep in mind is that in general solutions to elliptic, parabolic and hyperbolic equations behave very much like solutions to the Poisson, heat, and wave equations (with $f=0$ when comparing with properties of homogeneous equations). Because of this, we sometimes call the Poisson, heat, and wave equations the model equations.

elliptic equations: boundary value problems; Dirichlet or Neumann problems; mean value properties and maximum principle.

parabolic equations: Cauchy problems, initial-boundary value problems; infinity speed of propagation and smoothing properties; decay as $1/t^{n/2}$.

hyperbolic equations: Cauchy problems, initial-boundary value problems; domain of dependence/influence and finite propagation speed; decay as $1/t^{\frac{n-1}{2}}$.

We will not study these linear equations in detail here. But let us remark that the strategy to study them follow a pattern similar to what we used to study the model equations:

I. Without yet having proved existence, assume that a solution exists and derive some properties that a would-be solution must satisfy (e.g., D'Alemberts formula or the maximum principle). This step often goes by the name of a priori estimates (see below).

II. Use the knowledge (from I) about properties that solutions must have to actually construct solutions.

III. Study properties of solutions. This is in some sense similar to I, as we could imagine studying properties that solutions must have if they exist (without actually proving existence) The distinction here is one of depth: in I we want only as much information as needed to guide us toward a proof of existence, whereas here we want to understand as much as possible about solutions.

On the other hand, one of the main differences between the model equations and general linear equations (of one of the three types) is that for the former step I lead us to explicit formulas for what solutions should look like. In general, this is not possible, and instead in step I we devise the next best thing, which are a priori estimates. These are estimates that are valid for any solution of the equation if solutions exist (or any solution under certain assumptions, e.g., compactly supported data). They are called estimates instead of, say, identities or formulas because typically they are inequalities satisfied by solutions, if they exist.

Genally speaking, such a priori estimates provide us with enough information about solutions in order to guide us through an actual proof of existence. Examples of a priori estimates are:

- the maximum principle
- conservation of energy.

In these cases, we only used the fact that u was a solution, i.e., we did use the PDE, but did not use any formula for solutions. In fact, these results would remain true, as conditional statements, even if solutions turned out not to exist.

A priori estimates also play a large role in step III. Here, again, the goal is to obtain information about solutions even if explicit formulas are not available. An example was our decay estimate for the wave equation: we derived it without using explicit formulas for solutions. In fact, we could have proved it without knowing that solutions exist.

Finally, we remark that steps I, II, and III also provide a roadmap to the study of nonlinear PDEs.

We finish this section discussing the concept of <u>well-posedness</u> of a PDE. This concept was introduced by Hadamard. A problem (PDE, Cauchy problem, boundary value problem, etc.) is said to be well-posed if:

1 (Existence). The problem has a solution.

2 (Uniqueness). The problem admits no more than one solution.

3 (Continuous dependence on the data). Small changes in the equation or its "data" (e.g. initial data, boundary values, etc.) produces only small changes in the solution.

When talking about well-posedness relative to local solutions (e.g., solutions defined only for a short time) we use the term local well-posedness.

In practice, these concepts need to be made more precise in order to lead to well-defined problems (e.g., existence refers to classical, generalized, or some other type of solutions? How does one quantify small changes?) Nonetheless, these basic three concepts are at the core of PDE theory.

The method of characteristics

We are going to study the Cauchy problem for a first order quasilinear PDE in two variables (one spatial dimension), i.e.,

$$a(t,x,u)\,\partial_t u + b(t,x,u)\,\partial_x u + c(t,x,u) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}, \qquad (*)$$
$$u(0,x) = h(x).$$

We will employ the so-called method of characteristics, which roughly consists in transforming the PDE into a system of ODEs. Let us remark that this method is very general and can be applied to study equations of the form

$$F(Du, u, x) = 0 \quad \text{in } \Omega,$$
$$u = h \quad \text{on } \Gamma \subseteq \partial\Omega,$$

but the simple situation considered here will already capture the main ideas of the method.

We begin noticing that the PDE can be written as

$$(a, b, c)\cdot(\partial_t u, \partial_x u, 1) = 0.$$

Consider the graph of u. More precisely consider the parametric surface $g : (t,x) \in \mathbb{R}^2 \to (t, x, u(t,x)) \in \mathbb{R}^3$. A normal to the graph at $(t, x, u(t,x))$ can be written as

$$\partial_t g \times \partial_x g = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \partial_t u \\ 0 & 1 & \partial_x u \end{bmatrix} = -e_1 \partial_t u - e_2 \partial_x u + e_3 = (-\partial_t u, -\partial_x u, 1)$$

Hence:

$$0 = (a, b, c) \cdot (\partial_t u, \partial_x u, 1) = -(a, b, -c) \cdot (\partial_t g \times \partial_x g)$$

This means that $(a, b, -c)$ is tangent to the graph of u.

Thus, curves that have $(a, b, -c)$ as tangent vectors will lie on the graph of u, provided they start on the graph. We will use this fact to construct a family of curves that gives raise to a surface, which then will be showed to be the graph of u.

For each x_0 in $\{t=0\} \times \mathbb{R}$, we consider the system of ODEs:

$$\frac{dt}{d\tau} = a(t(\tau), x(\tau), u(\tau)),$$

$$\frac{dx}{d\tau} = b(t(\tau), x(\tau), u(\tau)),$$

$$\frac{du}{d\tau} = -c(t(\tau), x(\tau), u(\tau)),$$

for $(t(\tau), x(\tau), u(\tau))$ with initial condition at $\tau = 0$ given by

$$t(0) = 0, \quad x(0) = x_0, \quad u(0) = h(x_0).$$

The solution to this system is a curve $(t(\tau), x(\tau), u(\tau))$ in the (t, x, u) space (i.e., $\mathbb{R}^3$) parametrized by τ whose tangent vector is $(a, b, -c)$. This curve starts at $(0, x_0, h(x_0))$ which is the initial condition for our PDE at $t=0$, $x=x_0$. Because the point in the graph of u at $t=0$, $x=x_0$ is $(0, x_0, h(x_0))$, since $u(0,x) = h(x)$, the curve starts on the graph of u. It will, then, remain on the graph of u because $(a, b, -c)$ is tangent to the graph, as observed earlier.

If we consider a different point x_0, then we have a different curve. Thus, it is appropriate to write the system of ODEs and the solution curves as a system in the variable τ parametrized by α:

$$\begin{aligned}
\dot{t}(\tau,\alpha) &= a(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha)), \\
\dot{x}(\tau,\alpha) &= b(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha)), \\
\dot{u}(\tau,\alpha) &= -c(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha)),
\end{aligned} \qquad (**)$$

$$t(0,\alpha) = 0, \quad x(0,\alpha) = \alpha, \quad u(0,\alpha) = h(\alpha),$$

where $\dot{}$ is abbreviation for $\frac{d}{d\tau}$, i.e., $\dot{} = \frac{d}{d\tau}$.

The basic idea to consider this system of equations is that if we write

$$u = u(t,x) = u(t(\tau,\alpha), x(\tau,\alpha)) = u(\tau,\alpha)$$

then the chain rule gives

$$\frac{d}{d\tau} u(\tau,\alpha) = \partial_t u(t(\tau,\alpha), x(\tau,\alpha)) \overbrace{\dot{t}(\tau,\alpha)}^{= a(t(\tau,\alpha),\, x(\tau,\alpha),\, u(\tau,\alpha))}$$

$$+ \partial_x u(t(\tau,\alpha), x(\tau,\alpha)) \underbrace{\dot{x}(\tau,\alpha)}_{= b(t(\tau,\alpha),\, x(\tau,\alpha),\, u(\tau,\alpha))}$$

On the other hand

$$\frac{d}{d\tau} u(\tau,\alpha) = -c(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha)).$$

Therefore, we obtain that

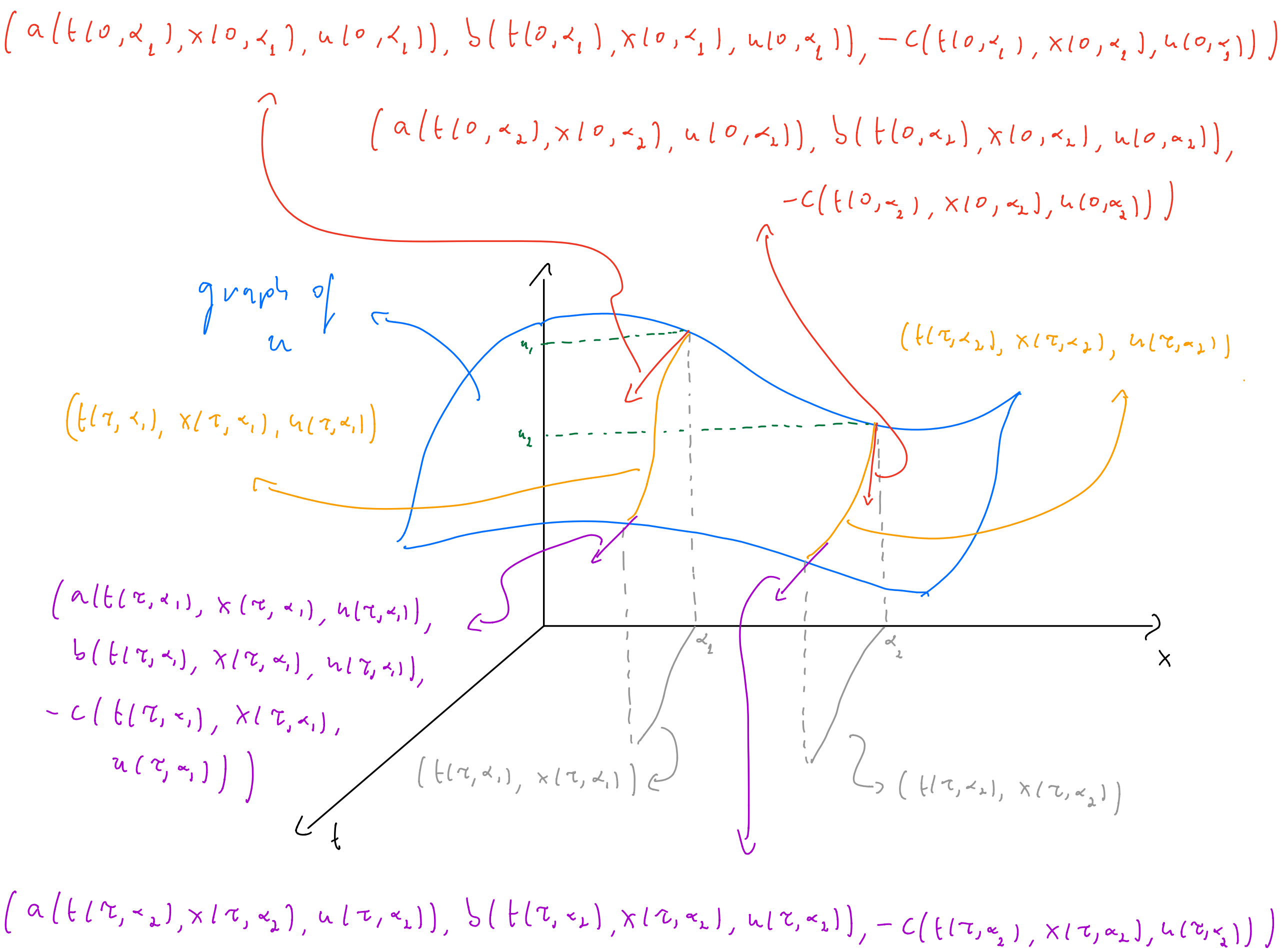
$$a\partial_t u + b\partial_x u + c = 0.$$

Moreover, we also have

$$u(0,x) = u(\tau(0,\alpha), x) = h(x).$$

We can also understand the system (**) in geometrical terms by considering the graph of u:

The graph of u is obtained by by taking the union of all $(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$ for different values of τ and α.



<u>Def</u>. The ODE system (**) is called the <u>characteristic system</u> (or system of characteristic equations) for the PDE (*). Its solutions $(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$ are called <u>characteristic curves</u>, or simply <u>characteristics</u>. The curves $(t(\tau,\alpha), x(\tau,\alpha))$ are called the <u>projected characteristic curves</u> or <u>projected characteristics</u>. We often abuse language and call $(t(\tau,\alpha), x(\tau,\alpha))$ the <u>characteristics or characteristic curves</u>.

Ex: Let us solve

$$\partial_t u + \partial_x u = 2$$
$$u(0,x) = x^2.$$

In this case $a = b = 1$, $c = -2$, so the characteristic system reads
$$\dot{t} = \dot{t}(\tau,\alpha) = 1, \quad \dot{x} = \dot{x}(\tau,\alpha) = 1, \quad \dot{u} = \dot{u}(\tau,\alpha) = 2.$$

The first equation gives $t(\tau,\alpha) = \tau + F(\alpha)$, where F is an unknown function of α. Using $t(0,\alpha) = 0$ we find $F(\alpha) = 0$. Next, $\dot{x} = 1$, gives $x(\tau,\alpha) = \tau + G(\alpha)$, where G is an unknown function of α. Using $x(0,\alpha) = \alpha$ we find $G(\alpha) = \alpha$. Finally, $\dot{u} = 2$ gives $u(\tau,\alpha) = 2\tau + H(\alpha)$, and $u(0,\alpha) = \alpha^2$ gives $u(\tau,\alpha) = 2\tau + \alpha^2$.

Hence
$$(t(\tau,\alpha),\ x(\tau,\alpha),\ u(\tau,\alpha)) = (\tau,\ \tau+\alpha,\ 2\tau+\alpha^2)$$
provides a parametric representation for the graph of u. To obtain u explicitly as a function of (t,x), we need to invert $(t(\tau,\alpha), x(\tau,\alpha))$ expressing $\tau = \tau(t,x)$ and $\alpha = \alpha(t,x)$. We find $\tau = t$, $\alpha = x - \tau = x - t$. Plugging into $u(\tau,\alpha)$ we find

$$u(t,x) = 2t + (x-t)^2.$$

Ex: Solve

$$3(t-1)^{2/3}\partial_t u + \partial_x u = 2,$$

$$u(0,x) = 1+x.$$

We have $a = 3(t-1)^{2/3}$, $b = 1$, $c = -2$, and

$$\dot{t} = 3(t-1)^{2/3}, \quad \dot{x} = 1, \quad \dot{u} = 2,$$

$$t(0,\alpha) = 0, \quad x(0,\alpha) = \alpha, \quad u(0,\alpha) = 1+\alpha.$$

We find:

$$\frac{dt}{3(t-1)^{2/3}} = d\tau \Rightarrow (t-1)^{1/3} = \tau + F(\alpha).$$

Since $t(0) = 0$, we find $F(\alpha) = -1$. Next, we find

$$x = \tau + \alpha, \quad u = 2\tau + 1 + \alpha.$$

Then $\tau = (t-1)^{1/3} + 1$, $\alpha = x - \tau = x - (t-1)^{1/3} - 1$, thus

$$u(t,x) = 2\left[(t-1)^{1/3} + 1\right] + 1 + x - (t-1)^{1/3} - 1$$

$$= (t-1)^{1/3} + x + 2.$$

Remark. The above two examples highlight the following aspects of the method characteristics:

I. To obtain $u = u(t,x)$, we need to invert the relations $t = t(\tau,\alpha)$ and $x = x(\tau,\alpha)$. Under which conditions is this map invertible?

II. Observe that the solution found in the second example is not differentiable at $t=1$, since $\partial_t u(t,x) = \frac{1}{3}\frac{1}{(t-1)^{2/3}}$. Hence, this solution is not defined for all time and we have obtained only a local solution. This is related to the fact that the coefficient of $\partial_t u$ in the PDE degenerates (i.e., becomes zero) at $t=1$. Alternatively, a point of view more in sync with the method of characteristics is the following:

III. Since we construct $u(t,x)$ out of a solution to the characteristic system, such a solution is defined only as long as $t(\tau,\alpha)$ and $x(\tau,\alpha)$ are defined. However, even though the PDE in the second example is linear, the characteristic system is a nonlinear system of ODEs (thus, the characteristic equations can be nonlinear even if the PDE is linear). We know from

ODE theory that nonlinear ODEs in general admit only local solutions. Therefore, we expect that the method of characteristics in general will produce only local solutions.

We now investigate the invertibility of the map $(\tau, \alpha) \mapsto (t(\tau,\alpha), x(\tau,\alpha))$. Write $\Phi(\tau,\alpha) = (t(\tau,\alpha), x(\tau,\alpha))$. For each (τ,α), if the Jacobian of Φ is nonzero at (τ,α) then the map Φ is invertible in a neighborhood of (τ,α). Compute

$$\text{Jacobian of } \Phi = J = \det \begin{bmatrix} \frac{\partial t}{\partial \tau} & \frac{\partial t}{\partial \alpha} \\ \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \alpha} \end{bmatrix}.$$

We consider the Jacobian $J = J(\tau,\alpha)$ for $\tau = 0$, for two reasons. First, as seen, we expect solutions to exist only locally, thus in general only for small values of τ. If we can show that $J(0,\alpha) \neq 0$ then, by continuity (assuming that we are dealing with sufficiently regular functions), we will also have $J(\tau,\alpha) \neq 0$ for small τ, guaranteeing the invertibility of Φ at least in a neighborhood of the initial surface $\{t=0\}$ (recall that $t(0,\alpha) = 0$). Second, in general we do not have much information about $\mathcal{J}$ (with exception of course of particular examples where we can compute $t(\tau,\alpha)$ and $x(\tau,\alpha)$ explicitly). However, as we will now see, at $\tau = 0$ we can relate J with the initial data.

From the characteristic system we have:

$$\frac{\partial t}{\partial \tau}(\tau,\alpha) = a(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha)),$$

so that, plugging $\tau = 0$:

$$\frac{\partial t}{\partial \tau}(0,\alpha) = a(t(0,\alpha), x(0,\alpha), u(0,\alpha))$$
$$= a(0, \alpha, h(\alpha)),$$

where we used that $t(0,\alpha) = 0$, $x(0,\alpha) = \alpha$, $u(0,\alpha) = h(\alpha)$. Since the functions a and h are given, we know what $\frac{\partial t}{\partial \tau}(0,\alpha)$ is.

Similarly we find

$$\frac{\partial x}{\partial \tau}(0,\alpha) = b(0, \alpha, h(\alpha)).$$

We also have that

$$\frac{\partial t}{\partial \alpha}(0,\alpha) = \frac{\partial t}{\partial \alpha}(\tau,\alpha)\Big|_{\tau=0} = \frac{\partial}{\partial \alpha}\left(t(\tau,\alpha)\Big|_{\tau=0}\right) = \frac{\partial}{\partial \alpha}\left(t(0,\alpha)\right) = 0 \text{ and}$$

$$\frac{\partial x}{\partial \alpha}(0,\alpha) = \frac{\partial x}{\partial \alpha}(\tau,\alpha)\Big|_{\tau=0} = \frac{\partial}{\partial \alpha}\left(x(\tau,\alpha)\Big|_{\tau=0}\right) = \frac{\partial}{\partial \alpha}\left(x(0,\alpha)\right) = 1,$$

where we used that $t(0,\alpha)=0$ and $x(0,\alpha)=\alpha$, and that for a C^1 function of two variables $f(w,z)$ we have

$$\frac{\partial}{\partial z} f(w,z)\Big|_{w=w_0} = \frac{\partial}{\partial z} f(w_0,z).$$

Therefore

$$J(0,\alpha) = \det \begin{bmatrix} a(0,\alpha,h(\alpha)) & 0 \\ b(0,\alpha,h(\alpha)) & 1 \end{bmatrix} = a(0,\alpha,h(\alpha)).$$

Hence, $J(0,\alpha)\neq 0$ whenever $a(0,\alpha,h(\alpha))\neq 0$. Note that this condition depends both on the coefficient a of the PDE and the initial data.

Def. The condition $J(0,\alpha)\neq 0$ is called the transversality condition. When the transversality condition holds we say that the Cauchy problem $(*)$ is non-characteristic.

Remark. The transversality condition in our case involves only $a(0,\alpha,h(\alpha))$ because of the simplifying choices we made at the beginning, i.e., to consider $t(0,\alpha)=0$, $x(0,\alpha)=\alpha$, and the data given along $\{t=0\}$. Recall that we mentioned that the method of characteristics is applicable to more general situations, and in these cases the transversality condition will be more involved.

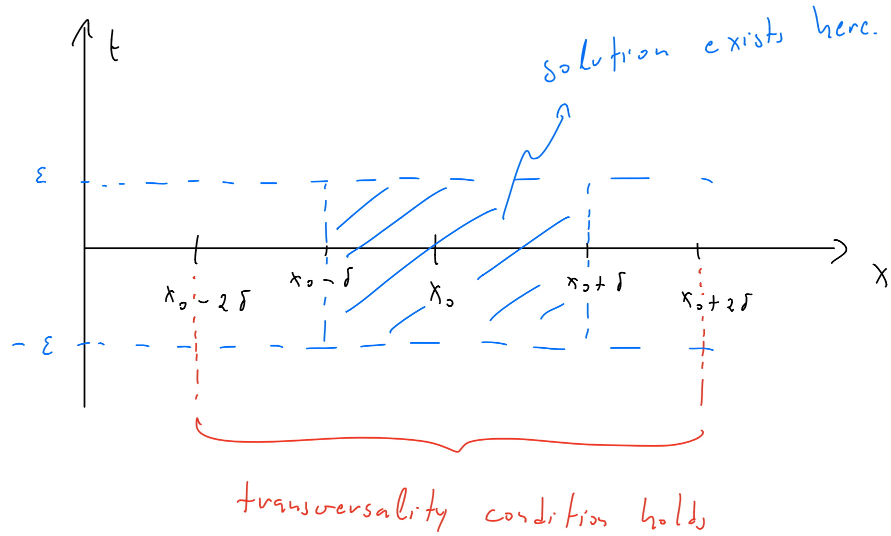
Theo. Consider the Cauchy problem

$$a(t,x,u)\,\partial_t u + b(t,x,u)\,\partial_x u + c(t,x,u) = 0 \quad \text{in } (0,\infty)\times\mathbb{R},$$
$$u(0,x) = h(x).$$

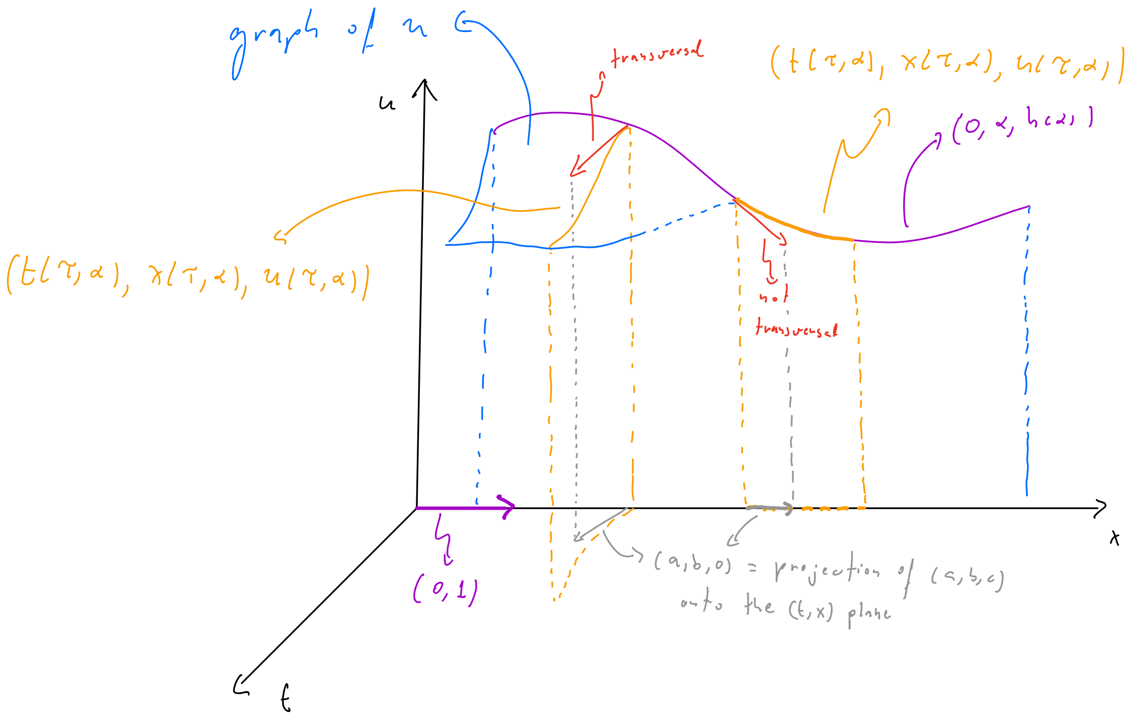
Assume that h is smooth and that a, b, and c are smooth functions of their arguments in a neighborhood of the initial curve $\{(0,x,h(x))\}_{x\in\mathbb{R}} \subset \mathbb{R}^3$. Let $x_0\in\mathbb{R}$ and suppose that there exists a $\delta>0$ such that the transversality condition holds for all x in the interval $(x_0-2\delta, x_0+2\delta)$. Then, there exists a $\varepsilon>0$ such that the above Cauchy problem admits a unique solution defined for $(t,x)\in(-\varepsilon,\varepsilon)\times(x_0-\delta, x_0+\delta)$. In particular, if the transversality condition holds for every $x\in\mathbb{R}$, then the Cauchy problem admits a unique solution defined in a neighborhood of $\{t=0\}\times\mathbb{R}$. If the transversality condition fails for every point on an interval $(A,B)\subseteq\mathbb{R}$, then the Cauchy problem has either no solution or infinitely many solutions.

Let's make some remarks before giving the proof.

Remark. Note that the solution is guaranteed to exist in a neighborhood that is smaller (in the x-direction) than where the transversality condition holds:



Remark. The intuition behind the theorem is the following. We want to find $u(t,x)$ by constructing the graph of u out of the curves $(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$. Such curves start on the portion of the graph of u corresponding to the initial data, i.e., $(0, \alpha, h(\alpha))$. We want to use the characteristic system to propagate the information on the initial curve to "inside" the graph of u. We do this by following the integral curves $(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$. This requires the tangent vectors to these curves to be transversal to $(0, \alpha, h(\alpha))$. If they are not, then the integral curves $(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$ cannot leave $(0, x, h(x))$ and move to the inside of the graph.



The vector $(a(0,\alpha,h(\alpha)), b(0,\alpha,h(\alpha)), c(0,\alpha,h(\alpha))$ will be transversal to $(0, \alpha, h(\alpha))$ if the vectors $(a(0,\alpha,h(\alpha)), b(0,\alpha,h(\alpha))$ and $(0,1)$ are linearly independent (see above picture). But this means precisely that

$$\det \begin{bmatrix} a(0,\alpha,h(\alpha)) & 0 \\ b(0,\alpha,h(\alpha)) & 1 \end{bmatrix} \neq 0,$$

which is the transversality condition.

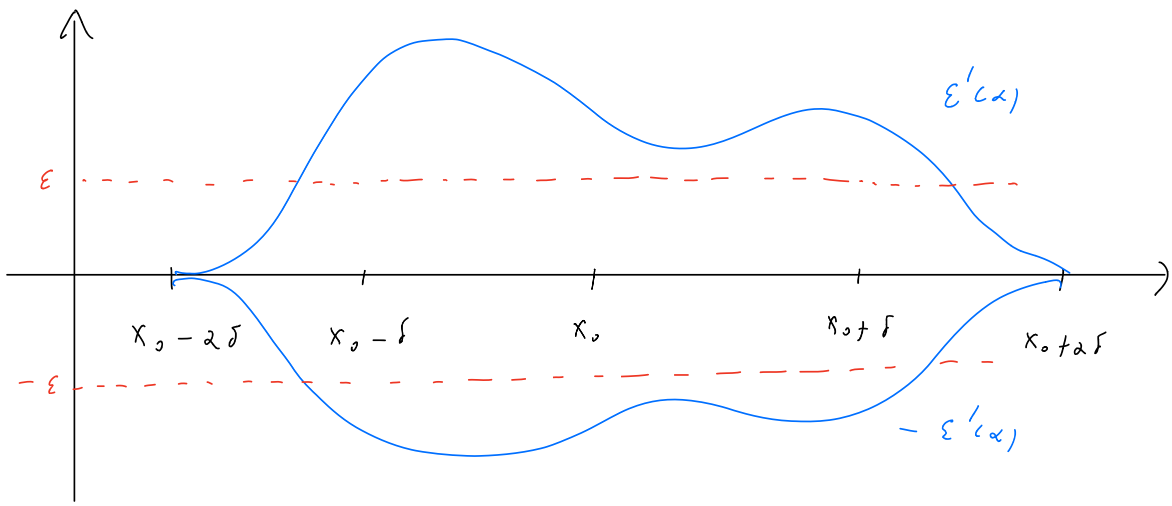
proof: Because the coefficients are smooth functions of its arguments, the existence and uniqueness theorem for ODEs guarantees that, for each point p on the initial curve $(0, \alpha, h(\alpha))$, there exists a unique characteristic curve starting at p. The union of these characteristic curves, i.e., image of the map

$$\Phi : (\tau, \alpha) \mapsto (t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$$

forms a parametric surface.

If the transversality condition holds, then the tangent vectors $\partial_\tau \Phi$ and $\partial_\alpha \Phi$ are linearly independent on the initial surface (since $\partial_\tau \Phi(0,\alpha) = (a(0,\alpha,h(\alpha)), b(0,\alpha,h(\alpha)), -c(0,\alpha,h(\alpha))$ and $\partial_\alpha \Phi(0,\alpha) = (0, 1, h'(\alpha))$). The existence and uniqueness theorem for ODEs also implies that Φ is a smooth function of τ and α. Therefore, by continuity, $\partial_\tau \Phi$ and $\partial_\alpha \Phi$ will remain linearly independent for $|\tau|$ sufficiently small, implying that Φ is a smooth non-degenerate (i.e., two-dimensional) parametric surface. For each α, we have an integral curve $\tau \mapsto (t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))$ defined for $|\tau| \leq \varepsilon'$, where $\varepsilon' > 0$ can depend on α, i.e.,

$\varepsilon' = \varepsilon'(\alpha)$. Invoking again the existence and uniqueness theorem for ODEs, we have that ε' varies continuously with α. Thus, if the transversality condition holds for $\alpha \in (x_0 - 2\delta, x_0 + 2\delta)$ and we consider the smaller interval $(x_0 - \delta, x_0 + \delta)$, we conclude that there exists a $\varepsilon > 0$ such that $\varepsilon'(\alpha) \geq \varepsilon$ for all $\alpha \in (x_0 - \delta, x_0 + \delta)$.



Notice that we can choose ε such that, for $(\tau, \alpha) \in (-\varepsilon, \varepsilon) \times (x_0 - \delta, x_0 + \delta)$, the map $(\tau, \alpha) \mapsto (t(\tau, \alpha), x(\tau, \alpha))$ is invertible.

Next, let us verify that the surface we constructed is indeed the graph of a function that solves the PDE. Set:

$$\tilde{u}(t, x) = u(\tau(t, x), \alpha(t, x))$$

for $(t, x) \in (\tau, \alpha)((-\varepsilon, \varepsilon) \times (x_0 - \delta, x_0 + \delta))$.

The chain rule gives:

$$\partial_t \tilde{u}(t,x) = \partial_\tau u(\tau(t,x)), \alpha(t,x)) \frac{\partial \tau}{\partial t} + \partial_\alpha u(\tau(t,x), \alpha(t,x)) \frac{\partial \alpha}{\partial t},$$

$$\partial_x \tilde{u}(t,x) = \partial_\tau u(\tau(t,x), \alpha(t,x)) \frac{\partial \tau}{\partial x} + \partial_\alpha u(\tau(t,x), \alpha(t,x)) \frac{\partial \alpha}{\partial x}.$$

Thus

$$a(t,x)\,\partial_t \tilde{u}(t,x) + b(t,x)\,\partial_x \tilde{u}(t,x)$$

$$= \partial_\tau u(\tau,\alpha) \Big(\underbrace{a(t,x) \frac{\partial \tau}{\partial t} + b(t,x) \frac{\partial \tau}{\partial x}} \Big)$$

$$+ \partial_\alpha u(\tau,\alpha) \Big(\underbrace{a(t,x) \frac{\partial \alpha}{\partial t} + b(t,x) \frac{\partial \alpha}{\partial x}} \Big).$$

But

$$\underline{1} = \partial_\tau \tau = \partial_\tau (\tau(t,x)) = \frac{\partial \tau}{\partial t} \underbrace{\frac{dt}{d\tau}}_{= a(t(\tau,\alpha),\, x(\tau,\alpha)) = a(t,x)} + \frac{\partial \tau}{\partial x} \overbrace{\frac{\partial x}{\partial \tau}}^{= b(t(\tau,\alpha),\, x(\tau,\alpha)) = b(t,x)} = \underbrace{a(t,x) \frac{\partial \tau}{\partial t} + b(t,x) \frac{\partial \tau}{\partial x}}$$

$$\underline{0} = \partial_\tau \alpha = \partial_\tau (\alpha(t,x)) = \frac{\partial \alpha}{\partial t} \frac{\partial t}{\partial \tau} + \frac{\partial \alpha}{\partial x} \frac{\partial x}{\partial \tau} = \underbrace{a(t,x) \frac{\partial \alpha}{\partial t} + b(t,x) \frac{\partial \alpha}{\partial x}}$$

hence

$$a(t,x)\,\partial_t \tilde{u}(t,x) + b(t,x)\,\partial_x \tilde{u}(t,x) = \partial_\tau u(\tau,\alpha)$$

$$= -c(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha)) = -c(t, x, \tilde{u}(t,x)),$$

showing the claim.

Now let us prove uniqueness. Say we have a smooth solution $v = v(t,x)$. In the region of interest we can write $t = t(\tau,\alpha)$ and $x = x(\tau,\alpha)$. Here, $(t(\tau,\alpha), x(\tau,\alpha))$ are the characteristic curves we have alredy constructed above, they solve the characteristic system with $a(t,x,u)$, $b(t,x,u)$, and $c(t,x,u)$ (and not $a(t,x,v)$ etc.). Put

$$\Psi(\tau,\alpha) = u(\tau,\alpha) - v(t(\tau,\alpha), x(\tau,\alpha)).$$

Because both u and v take the same initial data we have

$$\Psi(0,\alpha) = 0.$$

Differentiating with respect to τ:

$$\partial_\tau \Psi(\tau,\alpha) = \partial_\tau u(\tau,\alpha) - \partial_t v(t(\tau,\alpha), x(\tau,\alpha))\frac{\partial t}{\partial \tau} - \partial_x v(t(\tau,\alpha), x(\tau,\alpha))\frac{\partial x}{\partial \tau}.$$

$$= -c(\tau,\alpha,u(\tau,\alpha)) - a(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))\,\partial_t v(t(\tau,\alpha), x(\tau,\alpha))$$

$$- b(t(\tau,\alpha), x(\tau,\alpha), u(\tau,\alpha))\,\partial_x v(t(\tau,\alpha), x(\tau,\alpha)),$$

where we used the characteristic equations to replace $\dot{u}$, $\dot{t}$, and $\dot{x}$.

Since $u = \Psi + v$, we have:

$$\partial_\tau \Psi(\tau,\alpha) = -c(\tau,\alpha, v(\tau,\alpha) + \Psi(\tau,\alpha)) - a(\tau,\alpha, v(\tau,\alpha) + \Psi(\tau,\alpha))\,\partial_t v(\tau,\alpha)$$

$$- b(\tau,\alpha, v(\tau,\alpha) + \Psi(\tau,\alpha))\,\partial_x v(\tau,\alpha),$$

where we abbreviated $v(\tau,\alpha) = v(t(\tau,\alpha), x(\tau,\alpha))$, $\partial_t v(\tau,\alpha) = \partial_t v(t(\tau,\alpha), x(\tau,\alpha))$ etc. The above equation is, for each α, an ODE for Ψ with initial condition $\Psi(0,\alpha) = 0$. Since all functions on the RHS are smooth, this ODE admits a unique solution. Since v is a solution to the PDE,

$$a\partial_t v + b\partial_x v + c = 0,$$

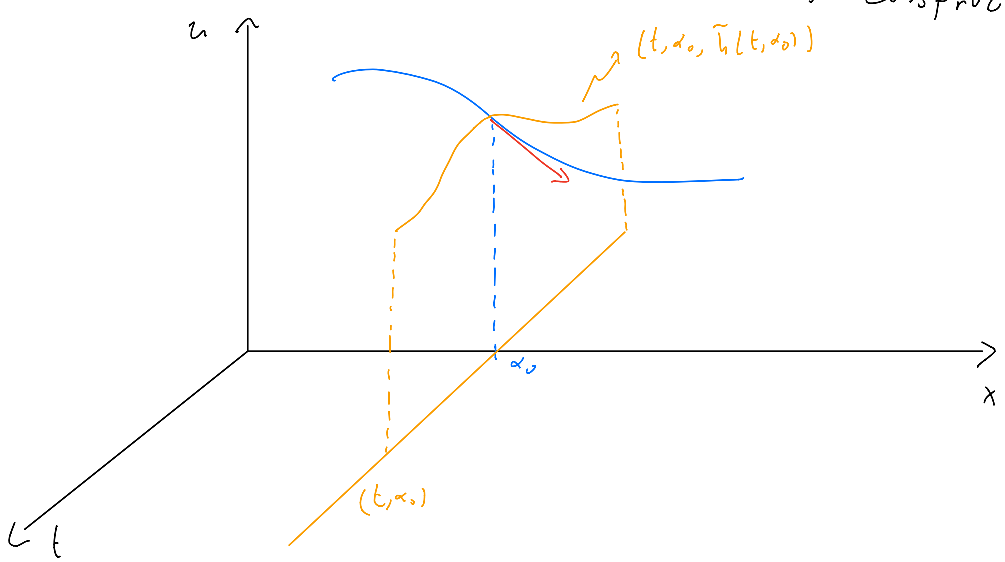
we see that $\Psi(t,\alpha) = 0$ is a solution to the ODE. By the ODE uniqueness, we obtain $u = v$.

Assume now that the transversality condition fails on an interval (A,B) as in the statement of the theorem. Then the characteristics $(t(\tau,\alpha), x(\tau,\alpha))$ lie on the x-axis (since (a,b) is parallel to $(0,1)$, see above discussion). The vector

$V = (a(0,\alpha,h(\alpha)), b(0,\alpha,h(\alpha)), -c(0,\alpha,h(\alpha))) = (0, b(0,\alpha,h(\alpha)), -c(0,\alpha,h(\alpha)))$,

$\alpha \in (A,B)$, is either tangent to the curve $(0,\alpha,h(\alpha))$ or it is not. If it is not, then there can be no solution. For, if a solution exists, we saw that $(a,b,-c)$ is tangent to the graph of the solution in particular it has to be tangent to $(0,\alpha,h(\alpha))$ for $\alpha \in (A,B)$.

If V is tangent to $(0, \alpha, h(\alpha))$, consider a line $x = \alpha_0$ where $\alpha_0 \in (A,B)$. Let $\tilde{h}(t,\alpha_0)$ be a smooth function on the line (t,α_0) such that $\tilde{h}(0,\alpha_0) = h(\alpha_0)$. Because $(0, b(0,\alpha,h(\alpha)))$ is transversal to the line (t,α_0), we can further choose $\tilde{h}$ such that the transversality condition holds on (t,α_0) in a neighborhood of $(0,\alpha_0)$. We can thus solve the Cauchy problem for our PDE with data given on (t,α_0) and the roles of t and x reversed. Since V is tangent to $(0,\alpha,h(\alpha))$, the characteristic curve starting on $(0,\alpha_0,\tilde{h}(0,\alpha_0)) = (0,\alpha_0,h(\alpha_0))$ is $(0,\alpha,h(\alpha))$. Thus, we obtain a solution to the PDE that takes the given data on $\{t=0\}\times(A,B)$. Clearly this solution is not unique in view of the many arbitrary choices we made to construct it.



$\square$

Further remarks on the method of characteristics

The method of characteristics can sometimes be used to study higher order equations. As an example, consider the wave equation

$$-u_{tt} + u_{xx} = 0,$$

$$u(0,x) = u_0(x)$$

$$u_t(0,x) = u_1(x)$$

Set $\quad v = u_t$ and $w = u_x$. Then

$$v_t = u_{tt} = u_{xx} = (u_x)_x = w_x$$

$$w_t = u_{xt} = u_{tx} = (u_t)_x = v_x$$

Thus, we can reduce the study of the wave equation to the study of the first-order system of PDEs:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_t \begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} v \\ w \end{pmatrix} = 0$$

$$v(0,x) = u_1(x),$$

$$w(0,x) = u_0'(x).$$

The method of characteristics can be generalized to certain systems of first-order PDEs. When we do so, the characteristic curves we find for the above system are precisely the characteristics of the wave equation as previously defined.

Arguing similarly to the above example, it is possible to show that any PDE can be written as a system of first order equations. This seems to suggest that any PDE can be treated with the method of characteristics. But although we can generalize the method certain systems of first order PDEs, not every first-order system can be treated by the method. Thus, the main application of the method of characteristics is to scalar first order equations.

Burgers' equation

We will now use the method of characteristics to study the Cauchy problem for Burgers' equation:

$$\partial_t u + u\,\partial_x u = 0, \qquad \text{in } (0,\infty)\times\mathbb{R}.$$
$$u(0,x) = h(x).$$

As a warm-up, let us begin studying the following linear equation

$$\partial_t u + c\,\partial_x u = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}$$
$$u(0,x) = h(x),$$

where c is a constant, known as transport equation.

The characteristic system reads:

$$\dot{t} = 1, \quad \dot{x} = c, \quad \dot{u} = 0,$$

which leads, using the initial conditions, to

$$t(\tau,\alpha) = \tau, \quad x(\tau,\alpha) = c\tau + \alpha, \quad u(\tau,\alpha) = h(\alpha).$$

Solving for (τ,α) in terms of (t,x) we find

$$u(t,x) = h(\alpha(t,x)) = h(x-ct).$$

This solution has a simple interpretation: consider a line $x - ct =$ constant, e.g., $x - ct = x_0$. Then, for any (t,x) along this line we have

$$u(t,x) = h(x-ct) = h(x_0).$$

Since the characteristics satisfy $x - ct = \alpha$, the line $x - ct = x_0$ is a characteristic with $\alpha = x_0$. Therefore, we conclude that u is constant along the characteristics, i.e., along the lines $x - ct =$ constant, with constant value determined by the initial condition. In particular, the derivative of u in the direction of a vector tangent to $x - ct = x_0$ must be zero. Considering the vector $(1,c)$, which is tangent to $x - ct = x_0$, we have

$$\underbrace{0}_{\text{because } u \text{ is constant in the } (1,c) \text{ direction}} = (1,c)\cdot\nabla u = (1,c)\cdot(\partial_t u, \partial_x u) = \partial_t u + c\,\partial_x u,$$

showing in another way that u satisfies the equation. Students should consider this simple example in mind for comparison when we consider Burgers' equation next.

For Burgers' equation, the characteristic system reads
$$\dot{t} = 1, \quad \dot{x} = u, \quad \dot{u} = 0.$$
The first and third equations give, using the initial conditions:
$$t(\tau, \alpha) = \tau, \quad u(\tau, \alpha) = h(\alpha).$$
Using u into the second equation and the initial condition $x(0, \alpha) = \alpha$ we find
$$x(\tau, \alpha) = \tau h(\alpha) + \alpha.$$
Inverting the above relations we find
$$\tau = \alpha, \quad \alpha = x - t h(\alpha(t, x))$$
But $u(t, x) = h(\alpha(t, x))$ so $\alpha = x - t u(t, x)$. We conclude that u is given in implicit form by
$$u(t, x) = h(x - t u(t, x)).$$
Compare with the solution to the transport equation where we had $c u_x$ instead of $u u_x$ in the PDE.

Consider a curve on the plane determined by the set of (t, x) such that
$$x - t u(t, x) = \text{constant},$$
e.g., let Γ_{x_0} be the curve determined by

$$x - t\,u(t,x) = x_0 .$$

Then, for (t,x) along Γ_{x_0} we have

$$u(t,x) = h(x_0),$$

so u is constant along this curve. Thus, along Γ_{x_0} we can also write $x - t\,u(t,x) = x_0$ as

$$x - t\,h(x_0) = x_0 .$$

Thus, we have that u is constant along the curve Γ_{x_0} given by

$$(t,\ t\,h(x_0) + x_0).$$

On the other hand, from the characteristic system we know that that the characteristic curves are given by

$$(t,\ t\,h(\alpha) + \alpha).$$

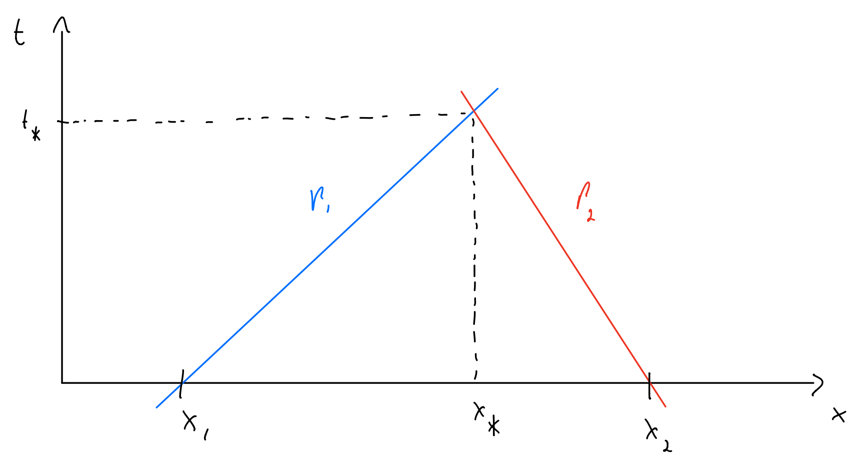
Comparing with the parametrization of Γ_{x_0} above, we conclude that Γ_{x_0} is a characteristic (with $\alpha = x_0$), and therefore $u(t,x)$ is constant along the characteriscs.

We will now explore an important consequence of this.

Shocks or blow-up of solutions for Burgers' equation.

We saw that both for the transport and Burgers' equation the solution is constant along the characteristics, which in both cases are straight lines. The main difference is that in the case of the transport equation all characteristics are parallel, i.e., they all have the same slope, whereas in the case of Burguers' equation different different characteristics can have different slopes since the slopes depend on $h(x)$. In particular, distinct characteristics might intersect for solutions of Burgers' equation.

What happens when characteristics intersect? Let us consider the following situation. Consider two characteristics Γ_1 and Γ_2 starting at $(0, x_1)$ and $(0, x_2)$, respectively, and suppose they intersect at (t_*, x_*):



We know that $u(t,x) = h(x_1)$ along Γ_1 and that $u(t,x) = h(x_2)$ along Γ_2. At (t_*, x_*) we must then have

$$h(x_1) = u(t_*, x_*) = h(x_2).$$

But h is a given function. In particular, h can be such that $h(x_1) \neq h(x_2)$, which would contradict the above equality. This suggest that in general u cannot be defined at (t_*, x_*), i.e., that something bad has to happen at the intersection of two characteristics.

Intuitively, we expect that a derivative of u must go to $\pm\infty$ at (t_*, x_*) – in the PDE jargon, we say that the solution <u>blows up</u> at (t_*, x_*) or forms a <u>shock-wave</u> (or <u>shock</u> for short). We expect that this is the case because u is trying to take two different values at (t_*, x_*), so it needs to do an "infinite jump" to do so. We assume throughout that h is C^∞, so u is C^∞ as long as it is defined.

Let us now see that shocks in fact can happen for solutions of burgers equation. Recall that the solution can be written in implicit form as

$$u(t,x) = h(x - t\,u(t,x)).$$

Differentiating:

$$\partial_x u(t,x) = h'(x - t\,u(t,x))\,(1 - t\,\partial_x u(t,x)).$$

Solving this relation for $\partial_x u$ gives

$$\partial_x u(t,x) = \frac{h'(x - t\,u(t,x))}{1 + t\,h'(x - t\,u(t,x))}.$$

The solution $u(t,x)$ is given by its constant value along a characteristic through (t,x). Along such a characteristic, we have $x - t\,u(t,x) = x_0$ for some constant value x_0 (see the previous discussion). Thus

$$\partial_x u(t,x) = \frac{h'(x_0)}{1 + t\,h'(x_0)}.$$

Therefore, we see that $|\partial_x u(t,x)| \longrightarrow \infty$ as $t \to -\dfrac{1}{h'(x_0)}$.

We call $t_* = -\frac{1}{h'(x_0)}$ a blow-up time.

Because we are considering only $t \geq 0$, a blow-up time will exist whenever $h'(x) < 0$ for some x. In particular, solutions with compactly supported data h will always blow up in finite time. Note that this has nothing to do with h being non-differentiable at some point, since h is a C^∞ function throughout. On the other hand, $\partial_x u$ does not blow up if $h'(x) \geq 0$ for every x (but notice that initial data of this type are exceptional).

We have not showed that the above blow-up is due to the intersection of the characteristics. So let us show that if characteristics do not intersect then no blow-up occurs.

Assume that we have a (smooth) solution u defined for $t < t_0$, and assume that no characteristics intersect up to time $t \leq t_0$. We will show that u is then a smooth solution in a neighborhood of $\{t = t_0\}$, in particular including values $t \geq t_0$. Being a smooth solution, u cannot blow-up in this neighborhood.

Since the solution is defined for $t < t_0$, the form of solutions we found implies that there is one (and only by the non-intersection hypothesis) characteristic through any $(t, x) \in \{t < 0\}$. (The fact that solutions in fact have the form we found follow from the uniqueness we have established.) Consider a point (t_0, x_0) and fix a $\delta > 0$. By assumption, no characteristic intersect along $\{t = t_0\} \times [x_0 - \delta, x_0 + \delta]$. It follows that characteristics cannot intersect in some neighborhood of (t_0, x_0) (non-intersecting is an "open condition"). For the characteristics in these neighborhood, u is defined by its constant value along the characteristics. This gives the claim since x_0 is arbitrary.

Scalar conservation laws in one dimension

Def. A quasilear PDE for a function $u = u(t,x)$, $(t,x) \in \Omega \subseteq \mathbb{R}^2$, that can be written as

$$\partial_t u + \partial_x (F(u)) = 0,$$

where $F : \mathbb{R} \to \mathbb{R}$ is a C^∞ map, is called a (scalar) conservation law in one (spatial) dimension.

Ex: Burgers' equation can be written as

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0,$$

so it is a conservation law with $F(u) = \frac{1}{2} u^2$.

Notice that a conservation law can be written as

$$\partial_t u + F'(u) \partial_x u = 0,$$

so they indeed correspond to quasilinear equations.

Remark. Conservation laws can be generalized to higher dimensions and to systems of PDEs, which we will study later. But the 1d case will already capture many of the main concepts.

In our discussion of the method of characteristics we saw that in general we expect that solutions to quasilinear equations will exist only for small times. Burgers' equation further illustrates that solutions can blow up in finite time. It is natural to ask whether it is possible to define the concept of solutions in a broader sense so that solutions to quasilinear equations can admit solutions (in this broader sense) that exist for all times, or at least for times longer than the blow-up time. For conservation laws, the answer is yes.

Def. A C^∞ function $\varphi : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ with compact support is called a test function. Let u be a bounded function such that $\int_\Omega u(t,x)\,dx\,dt$ and $\int_\Omega |u(t,x)|\,dx\,dt$ are well-defined for every bounded domain $\Omega \subset \mathbb{R}^2$. Let h be a function such that $\int_{\Omega'} h(x)\,dx$ and $\int_{\Omega'} |h(x)|\,dx$ are well defined for every bounded domain $\Omega' \subset \mathbb{R}$. We say that u is a weak solution to the Cauchy problem

$$\partial_t u + \partial_x (F(u)) = 0$$
$$u(0,x) = h(x)$$

if

$$\int_0^\infty \int_{-\infty}^\infty (u \partial_t \varphi + F(u) \partial_x \varphi) \, dx \, dt + \int_{-\infty}^\infty h(x) \varphi(0,x) \, dx = 0$$

for every test function φ.

Remark. Note that we do not require u to be defined everywhere in $[0,\infty) \times \mathbb{R}$. It only needs to be defined at "enough points" so that the integrals $\int_\Omega u \, dx \, dt$, $\int_\Omega |u| \, dx \, dt$ are defined. Similarly for h. (For students who took measure theory, we are saying that u and h are defined almost everywhere. And as u is bounded, we are saying $u \in L^\infty((0,\infty) \times \mathbb{R})$.)

Weak solutions are also called generalized solutions, integral solutions, or solutions in the sense of distributions. We use the term classical solution when we want to emphasize that a function u is a solution in the usual sense. We sometimes refer simply to solutions when the context makes it clear if we are talking about weak or classical solutions, or either.

The idea of weak solutions is the following. Suppose that u is a classical solution:

$$\partial_t u + \partial_x (F(u)) = 0 \quad \text{in } (0,\infty) \times \mathbb{R}$$

$$u(0,x) = h(x).$$

Multiply the equation by φ, where φ is a test function, and integrate over $(0,\infty) \times \mathbb{R}$:

$$\int_0^\infty \int_{-\infty}^{+\infty} \varphi \left(\partial_t u + \partial_x (F(u)) \right) dx\, dt = 0.$$

The integral is well-defined because φ has compact support. Integrating by parts an using again that φ has compact support,

$$- \int_0^\infty \int_{-\infty}^{\infty} \left(\partial_t \varphi\, u + \partial_x \varphi\, F(u) \right) dx\, dt$$

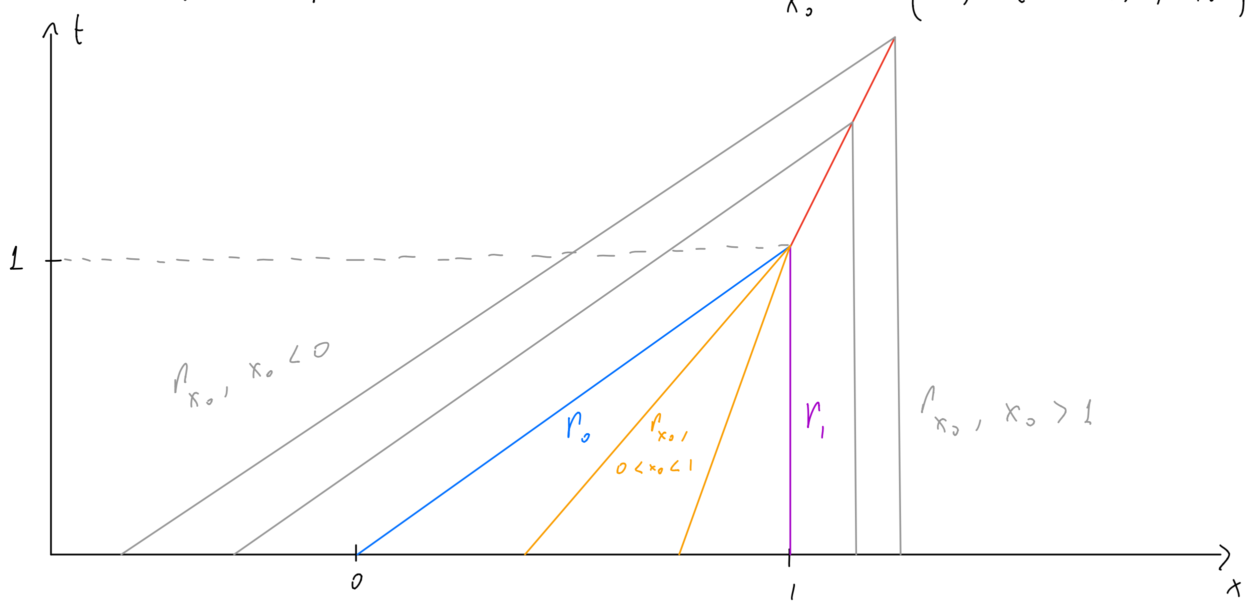
$$- \int_{-\infty}^{\infty} \varphi \overbrace{u \Big|_{t=0}}^{=h} dx = 0.$$

Since φ is an arbitrary test function, this shows that u is not only a classical solution but also a weak solution. So every classical solution is also a weak solution. Moreover, it will be a HW to show that if a weak solution is C^∞ and defined everywhere, then it is in fact a classical solution. The concept of weaker solution, however, is more general than that of a classical solution. Note that in the definition of weak solutions the function u does not even need to be differentiable.

<u>EX</u>: Consider Burgers' equation with data

$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Note that h is C^0 but not C^1. The characteristics of Burgers' equation are the lines $\Gamma_{x_0}(t) = (t, \, t\, h(x_0) + x_0)$.



For $x_0 \neq 0, 1$, h is smooth, so we can apply the method of characteristics for the curves starting at $x_0 \neq 0, 1$, and conclude that u is constant along these characteristics. Since the characteristics on the left and righ of $\Gamma_0(t)$ joint together at $\Gamma_0(t)$ for $t < 1$ (which is a consequence of the continuity of h), we see that u is also defined along $\Gamma_0(t)$ for $t < 1$. Similarly for $\Gamma_1(t)$.

Note that the characteristics intersect at $(1,1)$, so we know that something bad has to happen there. Writing u explicitly, we obtain (HW)

$$u(t,x) = \begin{cases} 1, & x \leq t, \ t < 1 \\ \dfrac{1-x}{1-t}, & t < x < 1, \ t < 1 \\ 0, & x \geq 1, \ t < 1. \end{cases}$$

Notice that indeed the solution becomes singular at $(1,1)$ (details discussed in a HW).

Let us now define a weak solution for $t \geq 1$. Since the characteristics are defined for $t > 1$, we can simply continue u by its constant value along the characteristics. More precisely, looking at the picture above we see that we can take $u = 1$ on the "left" and $u = 0$ on the right. This is defined except when the characteristics meet along the red line in the picture, which starts at $(1,1)$. Let $\Gamma_s(t) = (t, \beta t + 1 - \beta)$, which is a line passing through $(1,1)$, where $0 < \beta < 1$ is a parameter.

Set

$$u(t,x) = \begin{cases} 1 & , \; x < \beta t + 1 - \beta, \quad t \geq 1 \\ 0 & , \; x > \beta t + 1 - \beta, \quad t \geq 1. \end{cases}$$

Thus, u is defined everywhere except along $\Gamma_s(t)$, depicted in red in the picture.

Let us now test if u is a weak solution. We focus on $t \geq 1$, and note that away from Γ_s u is a classical solution. Let φ have support on Ω, where $\Omega \cap \{t \leq 1\} = \phi$ and $\Omega \cap \Gamma_s \neq \phi$. Then

$$\int_0^{\infty} \int_{-\infty}^{+\infty} \left(\partial_t \varphi \, u + \partial_x \varphi \, \frac{u^2}{2} \right) dx \, dt$$

$$= \int_{\Gamma_s} \left(\nu_t + \nu_x \frac{1}{2} \right) \varphi \, ds$$

where $\nu = (\nu_t, \nu_x)$ is the unit normal along Γ_s pointing to the right, and we used that $u = 1$ for $x < \beta t + 1 - \beta$ and $u = 0$ for $x > \beta t + 1 - \beta$, $t \geq 1$. ds is the element of integration along Γ_s.

Since $\Gamma_s(t) = (t, \beta t + 1 - \beta)$ we have that $\nu_t = -\beta / \sqrt{\beta^2 + 1}$ $\nu_x = 1 / \sqrt{\beta^2 + 1}$. Thus we get a weak solution if $\beta = 1/2$.

Rankine-Hugoniot conditions

We begin with a more precise definition of shocks.

Def. Let $r : (a,b) \to \mathbb{R}$ be a C^1 function and consider the C^1 curve $\Gamma(t) = \{ (t,x) \mid x = r(t) \}$. Let $u_r = u_r(t,x)$ and $u_l = u_l(t,x)$ be C^1 functions defined for $x \geq r(t)$ and $x \leq r(t)$, respectively. The function u defined by

$$u(t,x) = \begin{cases} u_l(t,x) & \text{for} \quad x < r(t), \\ u_r(t,x) & \text{for} \quad x > r(t) \end{cases}$$

is called a shock-wave or shock. The curve Γ is called a shock curve, although sometimes we also refer to Γ as the shock.

Remark. Note that the definition of a shock is independent of a conservation law PDE, but we are mostly interested in shocks that are weak solutions. Sometimes we emphasize this by using the term shock-solution.

Remark. The above definition can be generalized. E.g., we can consider multiple shock curves.

We now ask the following natural question: given a conservation law, under which conditions is a shock a (weak) solution? The answer is given in the next the next theorem.

Theo (Rankine-Hugoniot conditions). Let u be a shock with shock curve Γ. Then, u is a solution to the conservation law

$$\partial_t u + \partial_x (F(u)) = 0$$

if and only if

(a) $u = u(t,x)$ is a classical solution for $x \neq r(t)$.

(b) The Rankine-Hugoniot condition, defined by

$$F(u_r(t, r(t))) - F(u_\ell(t, r(t))) = r'(t)\,(u_r(t, r(t)) - u_\ell(t, r(t)))$$

holds on Γ.

proof. Let φ be a test function and Ω' a bounded domain containing the support of φ. Define the following sets:

$$\Omega := \Omega' \cap \{ (t,x) \mid t \geq 0 \},$$

$$\Omega_\ell := \Omega \cap \{ (t,x) \mid x < r(t) \},$$

$$\Omega_r := \Omega \cap \{ (t,x) \mid x > r(t) \}.$$

Then:

$$\int_0^{\infty}\int_{-\infty}^{\infty}(\partial_t\varphi\, u + \partial_x\varphi\, F(u))\,dt\,dx = \int_{\Omega}(\partial_t\varphi\, u + \partial_x\varphi\, F(u))\,dt\,dx$$

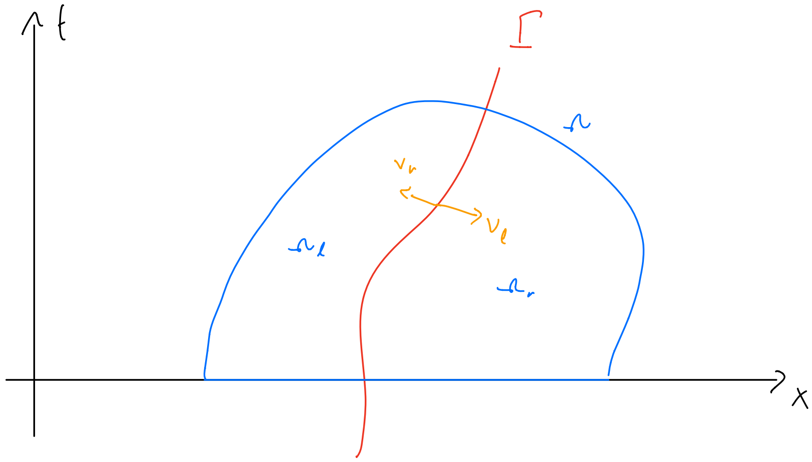
$$= \int_{\Omega_\ell}(\partial_t\varphi\, u + \partial_x\varphi\, F(u))\,dt\,dx + \int_{\Omega_r}(\partial_t\varphi\, u + \partial_x\varphi\, F(u))\,dt\,dx.$$

Using the fact that φ has support within Ω and that u is C^1 in Ω_ℓ, integration by parts produces

$$\int_{\Omega_\ell}(\partial_t\varphi\, u + \partial_x\varphi\, F(u))\,dt\,dx = -\int_{\Omega_\ell}\varphi(\partial_t u_\ell + \partial_x(F(u_\ell)))\,dt\,dx$$

$$+ \int_{\Gamma}\varphi(u_\ell \nu_\ell^t + F(u_\ell)\nu_\ell^x)\,ds - \int_{\Omega_\ell\cap\{t=0\}}\varphi\, u_\ell\, dx,$$

where $\nu_\ell = (\nu_\ell^t, \nu_\ell^x)$ is the unit outer normal to Ω_ℓ along Γ (so ν_ℓ points to the right, see picture below), and ds is the element of integration along Γ (see picture below).



Similarly,

$$\int_{\Omega_r} (\partial_t \varphi\, u + \partial_x \varphi\, F(u))\, dt\, dx = -\int_{\Omega_r} \varphi(\partial_t u_r + \partial_x(F(u_r)))\, dt\, dx$$

$$+ \int_{\Gamma} \varphi(u_r \nu_r^t + F(u_r)\nu_r^x)\, dt\, dx - \int_{\Omega_r \cap \{t=0\}} \varphi\, u_r\, dx,$$

where $\nu_r = (\nu_r^t, \nu_r^x)$ is the unit outer normal to Ω_r along Γ (so ν_r points to the left, see picture above).

Since $\Gamma(t) = (t, r(t))$, a tangent vector to Γ is given by $(1, \dot{r}(t))$, where $\dot{} = \frac{d}{dt}$. A normal vector $\nu = (\nu^t, \nu^x)$ to $(1, \dot{r}(t))$ satisfies

$$\nu^t + \dot{r}(t)\nu^x = 0, \quad \text{so} \quad \nu^t = -\dot{r}(t)\nu^x.$$

Then $|N| = \sqrt{(N^t)^2 + (N^x)^2} = |N^x| \sqrt{1 + (\dot{r}(t))^2}$. Thus, the vector

$$\frac{N}{|N|} = \frac{(-\dot{r}(t) N^x, N^x)}{|N|} = \frac{N^x}{|N^x|} \frac{1}{\sqrt{1 + \dot{r}^2}} (-\dot{r}, 1)$$

is normal to Γ and has unit length. Note that $N^x / |N^x| = \pm 1$. N points to the left if $N^x/|N^x| = -1$ and to the right if $= +1$. Thus

$$\nu_\ell = \frac{(-\dot{r}, 1)}{\sqrt{1 + \dot{r}^2}} \qquad \text{and} \qquad \nu_r = -\frac{(-\dot{r}, 1)}{\sqrt{1 + \dot{r}^2}}.$$

Therefore, we obtain:

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \varphi u + \partial_x \varphi F(u))\, dt\, dx = -\int_{\Omega_\ell} \varphi(\partial_t u_\ell + \partial_x(F(u_\ell)))\, dt\, dx$$

$$- \int_{\Omega_r} \varphi(\partial_t u_r + \partial_x(F(u_r)))\, dt\, dx - \int_{\Omega_\ell \cap \{t=0\}} \varphi u_\ell \, dx - \int_{\Omega_r \cap \{t=0\}} \varphi u_r \, dx$$

$$+ \int_\Gamma \varphi(-u_\ell \dot{r} + F(u_\ell)) \frac{ds}{\sqrt{1 + \dot{r}^2}}$$

$$- \int_\Gamma \varphi(-u_r \dot{r} + F(u_r)) \frac{ds}{\sqrt{1 + \dot{r}^2}}$$

$$= -\int_{\Omega_\ell} \varphi(\partial_t u_\ell + \partial_x(F(u_\ell)))\, dt\, dx - \int_{\Omega_r} \varphi(\partial_t u_r + \partial_x(F(u_r)))\, dt\, dx$$

$$- \int_{\Omega_\ell \cap \{t=0\}} \varphi\, u_\ell \, dx \; - \int_{\Omega_r \cap \{t=0\}} \varphi\, u_r \, dx$$

$$- \int_{\Gamma} \varphi \left((u_\ell - u_r)\dot{\gamma} + F(u_r) - F(u_\ell) \right) \frac{ds}{\sqrt{1+\dot{\gamma}^2}} .$$

Suppose that the Rankine - Hugoniot conditions hold. Then The first two integrals on the RHS above vanish since u_ℓ and u_r are classical solutions on Ω_ℓ and Ω_r, respectively, and the integral along Γ vanishes because (b) gives

$$F(u_r) - F(u_\ell) = \dot{\gamma}\,(u_r - u_\ell).$$

Thus

$$\int_0^{\infty} \int_{-\infty}^{+\infty} (\partial_t \varphi\, u + \partial_x \varphi\, F(u))\, dt\, dx$$

$$+ \int_{-\infty}^{+\infty} \varphi(x)\, u(0,x)\, dx = 0,$$

where we used that

$$- \int_{\Omega_\ell \cap \{t=0\}} \varphi\, u_\ell \, dx \; - \int_{\Omega_r \cap \{t=0\}} \varphi\, u_r \, dx = - \int_{-\infty}^{\infty} \varphi(x)\, u(0,x)\, dx .$$

Since φ is arbitrary, this shows that u is a weak solution. Reciprocally, if u is a weak solution, then

$$-\int_{\Omega_\ell} \varphi(\partial_t u_\ell + \partial_x(F(u_\ell)))\,dt\,dx \quad - \int_{\Omega_r} \varphi(\partial_t u_r + \partial_x(F(u_r)))\,dt\,dx$$

$$-\int_{\Gamma} \varphi\Big((u_\ell - u_r)\dot{\gamma} + F(u_r) - F(u_\ell) \Big) \frac{ds}{\sqrt{1+\dot{\gamma}^2}} =$$

$$\int_0^\infty \int_{-\infty}^{+\infty} (\partial_t \varphi\, u + \partial_x \varphi\, F(u))\,dt\,dx \quad + \int_{-\infty}^{+\infty} \varphi(x)\, u(0,x)\,dx$$

$$= 0$$

holds for every test function φ. Thus, we must have that u_ℓ and u_r are classical solutions in Ω_ℓ and Ω_r, respectively, and that $F(u_r) - F(u_\ell) = \dot{\gamma}(u_r - u_\ell)$ must hold along Γ. $\square$

<u>Ex</u>: It will be a HW to show that the weak solution to Burgers' equation constructed in a previous example satisfies the Rankine-Hugoniot conditions.

Notation. We denote

$$[[u]] = u_l - u_r = \text{"jump" in } u \text{ across } \Gamma$$

$$[[F(u)]] = F(u_l) - F(u_r) = \text{"jump" in } F \text{ across } \Gamma$$

$$\sigma = \dot{r}.$$

Then (b) in the Theorem reads

$$[[F(u)]] = \sigma [[u]].$$

Althoug the Rankine-Hugoniot conditions are (a) and (b), we often refers simply to (b), calling it "the" Rankine-Hugoniot condition.

Remark. As previously mentioned, the definition of shocks can be generalized. In particular, the definition can be extended to allow multiple shock curves, and the Rankine-Hugoniot conditions can also be generalized to this situation. We will often make use of these facts below.

Systems of conservation laws in one dimension

We will now generalize the study of conservation laws to systems.

Def. A system of conservation laws (in 1d) is a system of PDEs for $u = (u^1, u^2, \ldots, u^N)$, that can be written as

$$\partial_t u + \partial_x (F(u)) = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^2,$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is a C^∞ function.

Ex: The compressible Euler equations in fluid dynamics given by

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\partial_t (\rho v) + \partial_x (\rho v^2 + p) = 0$$

$$\partial_t \left(\rho \left(\tfrac{1}{2} v^2 + e \right) \right) + \partial_x \left(\rho v \left(\tfrac{1}{2} v^2 + e + \frac{p}{\rho} \right) \right) = 0,$$

Here, ρ is the density of the fluid, v the velocity, p the pressure, and e the internal energy. p is a known function of e and ρ. ρ, v, and e are the unknowns, which are functions of t and x. It will be a HW to show that the Euler system is a system of conservation laws.

__Remark.__ The definition of weak solutions, shocks, and the theorem on the Rankine-Hugoniot conditions generalize to systems of conservation laws. It will be a HW to do this generalization.

Using the chain rule, we can write

$$\partial_x (F(u)) = A(u) \partial_x u,$$

where $A(u)$ is a $N \times N$ matrix (depending on u). Thus, systems of conservation laws can be written

$$\partial_t u + A(u) \partial_x u = 0.$$

We turn our attention to these types of systems.

__Def.__ The system

$$\partial_t u + A(u) \partial_x u = 0$$

for $u = (u^1, \ldots, u^N)$, where $A = A(u)$ is a $N \times N$ matrix (depending on u) is a __strictly hyperbolic system__ if the matrix $A(u)$ admits N distinct real eigenvalues $\lambda_i = \lambda_i(u)$, which we order as

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_N(u).$$

We denote by $l = l(u)$ and $r = r(u)$ left and right eigenvectors of A, i.e.,

$$A(u)\,r(u) = \lambda(u)\,r(u), \quad l(u)\,A(u) = \lambda(u)\,l(u).$$

We say that a system of conservation laws is strictly hyperbolic if the corresponding system $\partial_t u + A(u)\partial_x u = 0$ is strictly hyperbolic.

Remark. Observe that the matrix $A(u)$ is simply the Jacobian matrix of F. I.e., if $F(u) = (F^1(u), \ldots, F^N(u)) = (F^1(u^1, \ldots, u^N), \ldots, F^N(u^1, \ldots, u^N))$, then

$$A(u) = \begin{bmatrix} \frac{\partial F^1}{\partial u^1} & \frac{\partial F^1}{\partial u^2} & \cdots & \frac{\partial F^1}{\partial u^N} \\ \frac{\partial F^2}{\partial u^1} & \frac{\partial F^2}{\partial u^2} & \cdots & \frac{\partial F^2}{\partial u^N} \\ \vdots & & & \\ \frac{\partial F^N}{\partial u^1} & \frac{\partial F^N}{\partial u^2} & \cdots & \frac{\partial F^N}{\partial u^N} \end{bmatrix}$$

Note that A always admits N linearly independent left and right eigenvectors by the assumption on the eigenvalues. We will denote by $\{l_i\}_{i=1}^N$ and $\{r_i\}_{i=1}^N$ sets of linearly independent eigenvectors.

Remark. We stress that the λ_i's, l_i's and r_i's depend on u since A does.

Remark. We will be discussing properties of solutions to systems of conservation laws, although we will not present an existence theory for such systems. But it is possible to develop tools (e.g., generalizations of the method of characteristics) to prove that such systems in general admit classical solutions.

Simple waves

Def. Let

$$\partial_t u + \partial_x (F(u)) = 0$$

be a strictly hyperbolic system of conservation laws. A C^1 simple wave in $\Omega \subset \mathbb{R}^2$ is a solution u of the form

$$u = \mathcal{U}(\psi(t,x)),$$

where $\psi : \Omega \to (a,b) \subseteq \mathbb{R}$ and $\mathcal{U} : (a,b) \to \mathbb{R}^N$ are C^1 functions. Similarly we can define C^k simple waves.

A simple wave has values on a curve (the image of $\mathcal{U}$), thus it can be thought as an intermediate case between constant solutions (values at a point) and general solutions (values on a surface).

Consider $u(t,x) = \mathcal{U}(\psi(t,x))$. Plugging into the equation:

$$\partial_t u + A(u)\partial_x u = \mathcal{U}'(\psi)\partial_t \psi + A(\mathcal{U}(\psi))\,\mathcal{U}'(\psi)\partial_x \psi$$

Suppose that $\mathcal{U}'(\psi)$ is an eigenvector of $A(\mathcal{U}(\psi))$,

$$A(\mathcal{U}(\psi))\,\mathcal{U}'(\psi) = \lambda(\mathcal{U}(\psi))\,\mathcal{U}'(\psi).$$

Then

$$\begin{aligned}\partial_t u + A(u)\partial_x u &= \mathcal{U}'(\psi)\partial_t \psi + \lambda(\mathcal{U}(\psi))\,\mathcal{U}'(\psi)\partial_x \psi \\ &= (\partial_t \psi + \lambda(\mathcal{U}(\psi))\partial_x \psi)\,\mathcal{U}'(\psi).\end{aligned}$$

u will be a solution if $\partial_t \psi + \lambda(\mathcal{U}(\psi))\partial_x \psi = 0$.

This argument provides us with the following method to look for simple wave solutions:

1. Find the eigenvalues $\lambda_i(u)$ and (right) eigenvectors $r_i(u)$ of $A(u)$, $i = 1, \dots, N$.

2. Find $\mathcal{U}_i(\tau)$ that solves the system of ODEs

$$\mathcal{U}_i'(\tau) = r_i(\mathcal{U}(\tau))$$

for some $i \in \{1, \dots, N\}$.

3. For an $i \in \{1, \dots, N\}$ for which step 2 was carried out, solve the scalar conservation law:

$$\partial_t \Psi + \lambda_i(\mathcal{U}_i(\Psi))\, \partial_x \Psi = 0$$

Then, $u(t,x) = \mathcal{U}_i(\Psi(t,x))$ is a simple wave solution.

The advantage of this method is that we solve a system of conservation laws by solving first a system of ODEs (step 2) and then a single equation conservation law (step 3).

Def. The solution $u(t,x) = \mathcal{U}_i(\Psi(t,x))$ described above is called a i-simple wave (i refers to the order $\lambda_1 < \dots < \lambda_N$ of the eigenvalues).

EX: Consider

$$\partial_t u + A(u)\, \partial_x u = 0$$

where $A(u)$ is given by

$$A(u) = \begin{pmatrix} u^2 & 0 \\ 0 & u^1 \end{pmatrix},$$

so the system reads

$$\begin{cases} \partial_t u^1 + u^2 \partial_x u^1 = 0, \\ \partial_t u^2 + u^1 \partial_x u^1 = 0. \end{cases}$$

Assume that $u^2(0,x) < u^1(0,x)$, so $u^2 < u^1$ for short time. The eigenvalues are $\lambda_1 = u^2 < \lambda_2 = u^1$, with eigenvectors $(1,0)$ and $(0,1)$, respectively. A 1-simple gives

$$\mathcal{U}_1'(\tau) = (1,0)$$

so u_2 is constant, and a 2-simple wave has u_1 constant. More details in this example will be given as a HW.

Rarefaction waves

Def. A *rarefaction wave* is a solution to the system

$$\partial_t u + A(u)\partial_x u = 0$$

with the following property:

(a) There exist $\alpha_\ell < \alpha_r$ and constant vectors $u_\ell, u_r \in \mathbb{R}^N$ such that $u = u_\ell$ for $x \leq \alpha_\ell t$ and $u = u_r$ for $x \geq \alpha_r t$.

(b) There exists a C^1 function $\mathcal{U} : [\alpha_\ell, \alpha_r] \to \mathbb{R}^N$ such that $\mathcal{U}(\alpha_\ell) = u_\ell$, $\mathcal{U}(\alpha_r) = u_r$, and

$$u(t,x) = \mathcal{U}\left(\frac{x}{t}\right)$$

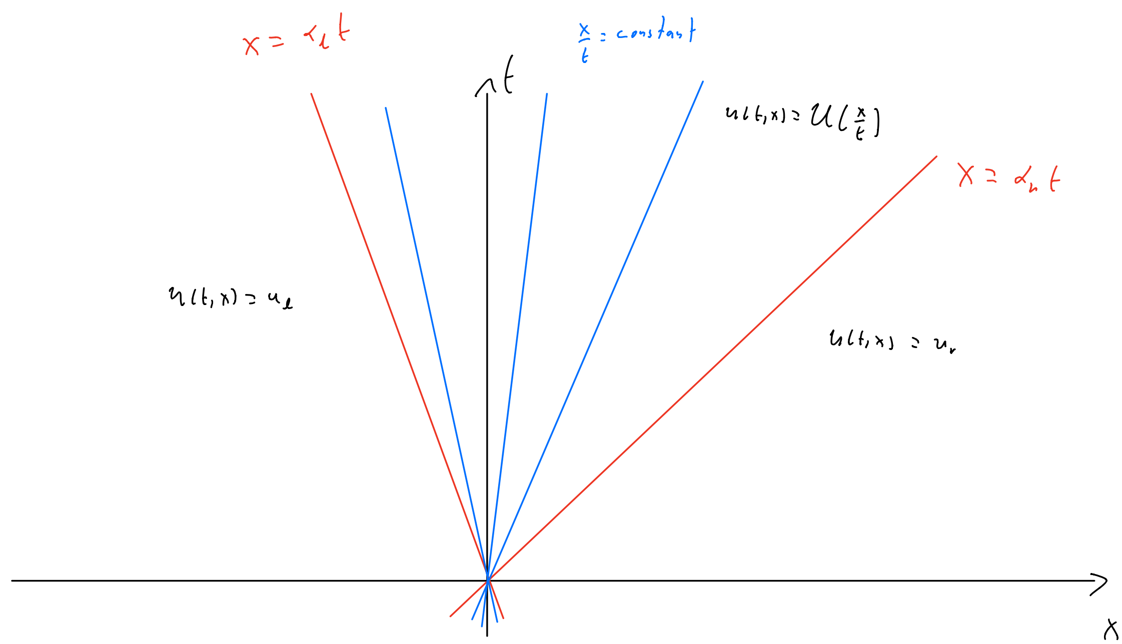
for $\alpha_\ell t < x < \alpha_r t$.

A rarefaction wave is a particular case of a simple wave, with

$$\psi(t,x) = \begin{cases} \alpha_\ell, & x \leq \alpha_\ell t, \\ x/t, & \alpha_\ell t < x < \alpha_r t. \\ \alpha_r, & \alpha_r t \leq x. \end{cases}$$

Note though that in general a rarefaction wave might fail to be C^1 across the lines $x = \alpha_\ell t$ and $x = \alpha_r t$, although it is a C^0 function (in particular, solutions here might mean weak solutions).

The picture below illustrates the behavior of rarefaction waves

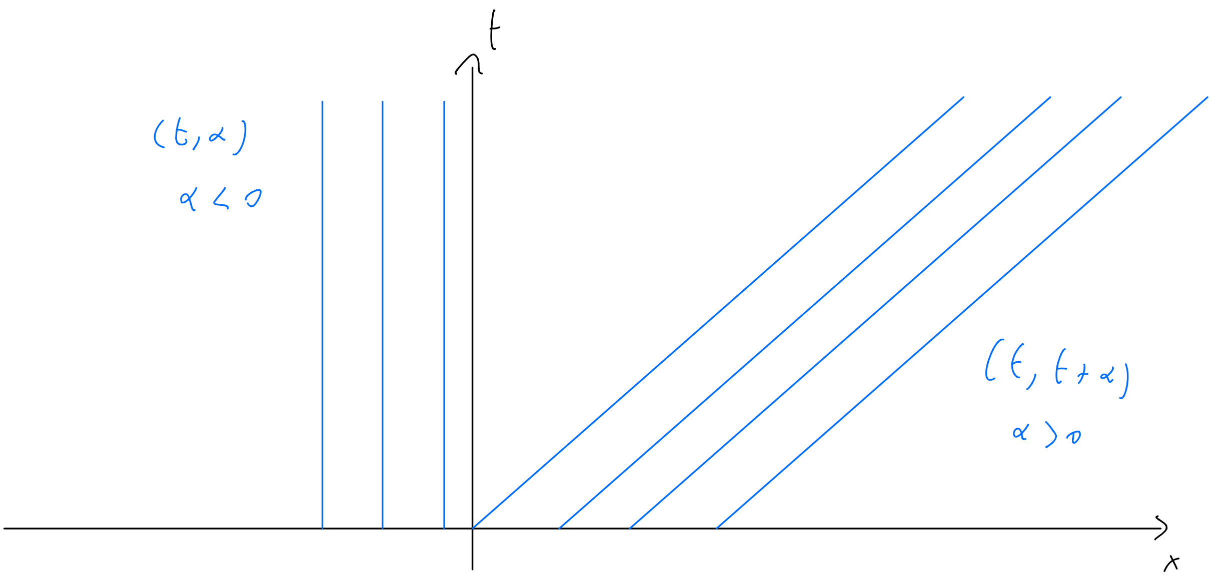


For any point on the line $x = \alpha t$, $\alpha_\ell < \alpha < \alpha_r$, we have $u(t,x) = \mathcal{U}(\frac{x}{t}) = \mathcal{U}(\alpha)$, thus u is constant along lines through the origin (since it is also constant along $x = \alpha t$ with $\alpha \leq \alpha_\ell$ or $\alpha \geq \alpha_r$).

Ex: Consider Burgers' equation with data

$$h(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

We have seen that the characteristics of Burgers' equation are $(t, x) = (t, h(\alpha)t + \alpha)$, $\alpha \in \mathbb{R}$. Therefore, the characteristics are (t, α) for $\alpha < 0$ and $(t, t+\alpha)$ for $\alpha > 0$.



The method of characteristics gives that u is constant along the characteristics, and in fact we get a classical solution in the region $x < 0$

or $x > t$, since u is in fact constant in these regions: $u(t,x) = 0$ for $x < 0$ and $u(t,x) = 1$ for $x > t$. However, the method does not give any information for $0 < x < t$, which is the region that is not reached by any of the characteristics (see picture above). If we define

$$u(t,x) = \begin{cases} 0, & x < 0, \\ \dfrac{x}{t}, & 0 < x < t, \\ 1, & x > t, \end{cases}$$

then we can verify that u satisfies the Rankine-Hugoniot conditions and, therefore, it is a weak solution to the problem. Moreover, u is a rarefaction wave.

It seems that there is a great deal of arbitrariness on how we obtained a weak solution in the above example. This is indeed the case. We will return to this point later on.

Let us now ask when can a rarefaction wave be a i-simple wave (in which case we refer to it as a i-rarefaction wave). For this, we need

$$\partial_t \Psi + \lambda_i(\mathcal{U}_i(\Psi))\partial_x \Psi = 0.$$

For $\alpha_\ell t < x < \alpha_r t$, we have $\Psi(t,x) = \frac{x}{t}$, so

$$-\frac{x}{t^2} + \lambda_i(\mathcal{U}_i(\Psi))\frac{1}{t} = 0,$$

thus $\lambda_i(\mathcal{U}_i(\Psi)) = x/t = \Psi(t,x)$. Moreover, since $\Psi(t,x) = \alpha_\ell$ for $x \leq \alpha_\ell t$ we must have $\lambda_i(u_\ell) = \alpha_\ell$. Similarly, $\lambda_i(u_r) = \alpha_r$. We conclude that $\lambda_i(\mathcal{U}(\tau)) = \tau$. In this case, we have

$$\frac{d}{d\tau}\lambda_i(\mathcal{U}_i(\tau)) = 1.$$

Using the chain rule and recalling that $\mathcal{U}'(\tau) = r_i(\mathcal{U}(\tau))$ for a i-simple wave, we have

$$\nabla\lambda_i(\mathcal{U}_i(\tau))\cdot r_i(\mathcal{U}_i(\tau)) = 1.$$

This is, therefore, a necessary condition for the existence of a rarefaction wave that is a i-simple wave. This motivates the following definition:

<u>Def.</u> The eigenvalue $\lambda_i(u)$ is said to be <u>genuinely nonlinear</u> if

$$r_i(u) \cdot \nabla \lambda(u) \neq 0.$$

In this case, r_i is said to be <u>normalized</u> if

$$r_i(u) \cdot \nabla \lambda(u) = 1.$$

Thus, having genuinely nonlinear eigenvalues is a necessary condition for the existence of i-rarefaction waves. The next theorem says that it is also sufficient.

<u>Thm.</u> Consider a strictly hyperbolic system of conservation laws

$$\partial_t u + \partial_x (F(u)) = 0,$$

and suppose that for some i the eigenvalue $\lambda_i(u)$ is genuinely nonlinear. Then, there exists a i-rarefaction wave solution to the system.

proof. Let $u_\ell \in \mathbb{R}^N$ be constant and define

$$\alpha_\ell = \lambda_i(u_\ell).$$

Let U_i be a solution to the ODE

$$U_i'(\tau) = r_i(U_i(\tau)),$$
$$U_i(\alpha_\ell) = u_\ell.$$

Let $\alpha_r > \alpha_\ell$ be such that $U_i(\alpha_r)$ is defined and set

$$u_r = U_i(\alpha_r).$$

We can assume that $r_i(u)$ is normalized. Then

$$\frac{d}{d\tau}\lambda_i(U_i(\tau)) = U_i'(\tau)\cdot\nabla\lambda_i(U_i(\tau)) = r_i(U_i(\tau))\cdot\nabla\lambda_i(U_i(\tau)) = 1.$$

This implies that $\lambda_i(U_i(\tau)) = \tau + \text{constant}$. Because $U(\alpha_\ell) = u_\ell$ and $\lambda_i(u_\ell) = \alpha_\ell$, the constant is zero and thus $\lambda_i(U_i(\tau)) = \tau$. Define

$$u(t,x) = \begin{cases} u_\ell, & x \le \alpha_\ell t, \\ U_i(\frac{x}{t}), & \alpha_\ell t < x < \alpha_r t, \\ u_r, & x \ge \alpha_r t. \end{cases}$$

Consider the region $\alpha_\ell t < x < \alpha_r t$. Since U_i satisfies $U_i'(\tau) = r_i(U_i(\tau))$, U_i verifies step 2 of our three-step recipe for the construction of simple wave solutions. Moreover, since $\lambda_i(\mathcal{U}_i(\tau)) = \tau$, we have, with $\psi(t,x) = \frac{x}{t}$,

$$\partial_t \psi + \underbrace{\lambda_i(\mathcal{U}_i(\psi))}_{=\psi} \partial_x \psi = -\frac{x}{t^2} + \frac{x}{t} \cdot \frac{1}{t} = 0,$$

so we have verified step 3 of the recipe as well. Thus u is a solution for $\alpha_\ell t < x < \alpha_r t$. For $x < \alpha_\ell t$ and $x > \alpha_r t$, u is constant so it is trivially a solution. Finally, along $x = \alpha_\ell t$ and $x = \alpha_r t$, the limits from both sides agree, i.e., u is continuous. Thus, the jump in u and in $F(u)$ vanish and the Rankine-Hugoniot conditions are satisfied.

$\square$

Riemann's problem

The Riemann problem consists of the following: find a solution to the system of conservation laws

$$\partial_t u + \partial_x (F(u)) = 0 \quad \text{in } (0,\infty) \times \mathbb{R}$$

with initial data

$$u(0,x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0, \end{cases}$$

where $u_\ell, u_r \in \mathbb{R}^N$ are constant vectors.

We saw above how to construct solutions that are rarefaction waves. Since such solutions satisfy $u(0,x) = u_\ell$ for $x < 0$ and $u(0,x) = u_r$ for $x > 0$, they are natural candidates for solutions to the Riemann problem. But it is important to notice that our previous theorem does not automatically give a solution to Riemann's problem because in the latter u_ℓ and u_r are given, whereas in our construction of rarefaction waves we were free to choose u_ℓ but not u_r. Indeed, recall that u_r was determined by choosin α_r

and setting $u_r = \mathcal{U}_i(\alpha_r)$. Therefore, in the case of the Riemann problem, we need that u_r is in the image of $\mathcal{U}_i$. This motivates the following definition.

<u>Def.</u> For a given strictly hyperbolic system of conservation laws, let $\mathcal{U}_i$ be in our discussion of i-rarefaction waves. Consider the curve $\mathcal{U}_i(\tau)$ in $\mathbb{R}^N$. Given $z_0 \in \mathbb{R}^N$, we denote the curve $\mathcal{U}_i(\tau)$ by $\mathcal{R}_i(z_0)$ if it passes through z_0, and call it the <u>i^{th}-rarefaction curve</u>. We set

$$\mathcal{R}_i^+(z_0) := \{ z \in \mathcal{R}_i(z_0) \mid \lambda_i(z) > \lambda_i(z_0) \}$$

$$\mathcal{R}_i^-(z_0) := \{ z \in \mathcal{R}_i(z_0) \mid \lambda_i(z) < \lambda_i(z_0) \}$$

so that
$$\mathcal{R}_i(z_0) = \mathcal{R}_i^-(z_0) \cup \{z_0\} \cup \mathcal{R}_i^+(z_0).$$

<u>Theo.</u> Consider the Riemann problem and suppose that for some i the eigenvalue λ_i is genuinely nonlinear and that $u_r \in \mathcal{R}_i^+(u_\ell)$. Then, there exists a (weak) solution to the Riemann problem. This solution is a i-rarefaction wave.

proof. The proof is essentially contained in the proof of the previous theorem. We just need to verify that the additional assumption $u_r \in R_i^+(u_\ell)$ gives us what we want.

Recall that we had set $\alpha_\ell = \lambda_i(u_\ell)$ (where u_ℓ was arbitrary in the previous proof but here it is given by the initial condition), and solved

$$\mathcal{U}_i'(\tau) = r_i(\mathcal{U}_i(\tau)),$$

$$\mathcal{U}_i(\alpha_\ell) = u_\ell.$$

We now claim that if $z \in R_i^+(u_\ell)$, then $z = \mathcal{U}_i(\alpha)$ for some $\alpha > \alpha_\ell$ (note that by definition $z \neq u_\ell$). Set $\alpha = \lambda(z)$ and solve the ODE for $\mathcal{U}_i$ with initial condition $\mathcal{U}_i(\alpha) = z$. ODE uniqueness guarantees that the solution starting at u_ℓ passes through z, and $\alpha > \alpha_\ell$ since $z \in R_i^+(u_\ell)$. Thus, there exists $\alpha_r > \alpha_\ell$ such that $u_r = \mathcal{U}_i(\alpha_r)$. The rest of the proof is as in the previous theorem.

$\square$

Riemann invariants

Def. A C^1 function $R^i : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is called an *i-Riemann invariant* for the strictly hyperbolic system

$$\partial_t u + A(u)\partial_x u = 0 \quad \text{in } \Omega$$

if $\nabla R^i(z) \cdot r_i(z) = 0, \quad z \in \Omega.$

Thus, R^i is constant along the integral curves of r_i.

Let us make the following remark, which we will need further below: We have $r_i \cdot l_j = 0$ for $i \neq j$. To see it,

$$\left.\begin{array}{l} l_j \cdot (A r_i) = \lambda_i \, l_j \cdot r_i \\ \quad\; \| \\ (l_j A) \cdot r_i = \lambda_j \, l_j \cdot r_i \end{array}\right\} \Rightarrow (\lambda_i - \lambda_j)\, l_j \cdot r_i = 0 \Rightarrow l_j \cdot r_i = 0$$

since $\lambda_i \neq \lambda_j$ for $i \neq j$.

In particular, ∇R^i is parallel to l_j, $j \neq i$, for 2×2 systems. It follows that Riemann invariants always exist for 2×2 systems. To see this, consider the system

$$\partial_t u + A(u)\partial_x u = 0.$$

Letting $r = [r_1\ r_2]$ be the matrix whose columns are the eigenvectors r_1, r_2, we have

$$r^{-1} A r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

so $A = r \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} r^{-1}$, and we can write

$$\partial_t u + r \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} r^{-1} \partial_x u = 0,$$

or yet

$$(r^{-1})\partial_t u + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} r^{-1} \partial_x u = 0$$

In components, with the matrix convention $(\cdot)^{\text{row}}{}_{\text{column}}$

$$(r^{-1})^i{}_j \partial_t u^j + \lambda_i (r^{-1})^i{}_j \partial_x u^j = 0 \quad (\text{no sum over } i).$$

Writing the integral curves of r_i as $(t, x_i(t))$, where $\frac{dx_i}{dt} = \lambda_i$, we have that

$$\frac{d}{dt} u^j(t, x(t)) = \partial_t u^j(t, x(t)) + \lambda_i \partial_x u^j(t, x(t))$$

so that we can write the equation as

$$(r^{-1})^i{}_j \frac{d}{dt} u^j = 0.$$

Now we look for a function $\Phi(u)$ such that

$$\Phi_i(u)(r^{-1})^i{}_j \frac{d}{dt} u^j = \frac{d}{dt} R^i \qquad (\text{no sum over } i)$$

for some R^i; notice that then this R^i will be an i-Riemann invariant since $\frac{d}{dt} R^i = \partial_t R^i + \lambda_i \partial_x R^i = 0$ (no sum over i), i.e., R^i is constant along the characteristics. We write the desired equality in differential form

$$\Phi_i(u)(r^{-1})^i{}_j \, du^j = dR^i. \qquad (\text{no sum over } i)$$

This means that Φ_i is an integrating factor for $(r^{-1})^i{}_j \, du^j$. From ODE theory, we know such an integrating factor always exists; this is the point where we are explicitly using that the system is 2×2.

Remark. For $N\times N$ systems, $N \geq 3$, Riemann invariants do not always exist.

Riemann invariants are particularly useful for 2×2 systems:

$$\partial_t u^1 + \partial_x \left(F^1(u^1,u^2)\right) = 0 \quad \text{in } (0,\infty)\times\mathbb{R},$$
$$\partial_t u^2 + \partial_x \left(F^2(u^1,u^2)\right) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}$$
$$u^1(0,x) = h^1(x),$$
$$u^2(0,x) = h^2(x),$$

or, in compact form,

$$\partial_t u + \partial_x \left(F(u)\right) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}$$
$$u(0,x) = h(x),$$

$u = (u^1, u^2)$, $F = (F^1, F^2)$, $h = (h^1, h^2)$.

For a given 2×2 system with Riemann invariants, let us assume that the map

$$\Phi(R^1, R^2) = \left(R^1(z^1,z^2), R^2(z^1,z^2)\right)$$

is a diffeomorphism. Set

$$v(t,x) = \Phi(u(t,x))$$

for $u(t,x)$ a solution to the above system. Then, $v = (v^1, v^2)$ satisfies

$$\partial_t v^1 + \Lambda_2(v)\,\partial_x v^1 = 0,$$
$$\partial_t v^2 + \Lambda_1(v)\,\partial_x v^2 = 0,$$

where Λ_i is the eigenvalue λ_i expressed in terms of v, i.e.,

$$\Lambda_i(v) = \lambda_i(\Phi^{-1}(v)).$$

The equations for v follow from the following computation: for $i \neq j$, we have

$$\begin{aligned}
\partial_t v^i + \Lambda_j(v)\,\partial_x v^i &= \partial_t v^i + \lambda_j(u)\,\partial_x v^i \\
&= \nabla R^i(u)\cdot\partial_t u + \lambda_j(u)\,\nabla R^i(u)\cdot\partial_x u \\
&= (\partial_t u + \lambda_j(u)\,\partial_x u)\cdot\nabla R^i(u) \\
&= (-\partial_x(F(u)) + \lambda_j(u)\,\partial_x u)\cdot\nabla R^i(u) \\
&= ((-DF(u) + \lambda_j(u)\,I)\,\partial_x u)\cdot\nabla R^i(u),
\end{aligned}$$

where DF is the Jacobian matrix of F and I is the identity matrix. We can also write the above as

$$\left(\nabla R^i(u)\left(-A(u) + \lambda_j(u) I\right)\right)\partial_x u.$$

Since $\nabla R^i = 0$ along the integral curves of r_i, ∇R^i is parallel to l_j, thus ∇R^i is a left eigenvector with eigenvalue λ_j and the term in parenthesis vanishes.

Observe that v_i is constant along the integral curve $(t, x_i(t))$ where $\frac{dx_i}{dt} = \lambda_i(u(t, x(t)))$ since

$$\frac{d}{dt} v_i(t, x_i(t)) = \partial_t v_i(t, x_i(t)) + \lambda_i(u(t, x(t)))\partial_x v(t, x(t)).$$

<u>Theo.</u> Assume that the system

$$\partial_t u^1 + \partial_x(F^1(u^1,u^2)) = 0 \quad \text{in } (0,\infty)\times\mathbb{R},$$
$$\partial_t u^2 + \partial_x(F^2(u^1,u^2)) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}$$
$$u^1(0,x) = h^1(x),$$
$$u^2(0,x) = h^2(x),$$

is strictly hyperbolic and that the eigenvalues λ_i, $i=1,2$, are genuinely nonlinear. Assume that h has compact support. Let $R=(R^1,R^2)$ be Riemann invariants for the system and assume that $\nabla R^i \neq 0$, $i=1,2$. Set $v = \Phi(u)$ as above (which is well-defined, see below). If either $\partial_x v^1 < 0$ or $\partial_x v^2 < 0$ somewhere in $\{t=0\}\times\mathbb{R}$, then the system cannot have a smooth solution u that exists for all $t\geq 0$.

<u>proof.</u> The assumption $\nabla R^i \neq 0$ implies that $(R^1(z^1,z^2), R^2(z^1,z^2))$ define a system of coordinates in $\mathbb{R}^2$ (via the level sets of R^i). In particular, $v = \Phi(u)$ is well-defined.

Consider $\lambda_i = \lambda_i(z^1, z^2)$ as a function of (R^1, R^2), i.e., $\lambda_i(z^1, z^2) = \lambda_i(z^1(R^1,R^2), z^2(R^1,R^2))$. Then

$$\frac{\partial \lambda_i}{\partial R^j} = \frac{\partial \lambda_i}{\partial z^k}\frac{\partial z^k}{\partial R^j}.$$

We also have that

$$\frac{\partial R^i}{\partial z^k}\frac{\partial z^k}{\partial R^j} = \frac{\partial R^i}{\partial R^j} = \delta^i_j.$$

Hence, for $i \neq j$, $\frac{\partial}{\partial R^j} z = \frac{\partial}{\partial R^j}(z^1, z^2)$ is perpendicular to $\nabla R^i(z)$. But $\nabla R^i(z)$ is perpendicular to $r_i(z)$, thus $\frac{\partial}{\partial R^j} z$ is parallel to r_i, $i \neq j$. Thus $\frac{\partial z}{\partial R^j} = f r_i$ for some $f \neq 0$. Hence

$$\frac{\partial \lambda_i}{\partial R^j} = f \frac{\partial \lambda_i}{\partial z^k}(r_i)^k = f \nabla \lambda_i \cdot r_i.$$

But $\nabla \lambda_i \cdot r_i \neq 0$ by our assumption that the eigenvalues are genuinely nonlinear, so this assumption can equivalently be stated as

$$\frac{\partial \lambda_i}{\partial R^j} \neq 0, \quad i \neq j$$

If u is a smooth solution to the system, we set

$$a := \partial_x v^1 \quad \text{and} \quad b := \partial_x v^2.$$

Note that we have already showed that $v = (v^1, v^2)$ solves

$$\partial_t v^1 + \lambda_2(v)\,\partial_x v^1 = 0,$$
$$\partial_t v^2 + \lambda_1(v)\,\partial_x v^2 = 0.$$

Differentiate the first equation with respect to x to obtain:

$$\partial_t a + \lambda_2 \partial_x a + \frac{\partial \lambda_2}{\partial R^i} \frac{\partial v^i}{\partial x} \partial_x v^1 = 0,$$

$$\partial_t a + \lambda_2 \partial_x a + \frac{\partial \lambda_2}{\partial R^1} a^2 + \frac{\partial \lambda_2}{\partial R^2} ab = 0$$

Adding and subtracting $\lambda_2 \partial_x v^2 = \lambda_2 b$ to the v^2 equation

$$\partial_t v^2 + \lambda_2 \partial_x v^2 - (\lambda_2 - \lambda_1)\, b = 0.$$

Solving for b and plugging into the $\partial_t a$ equation (recall that $\lambda_2 - \lambda_1 \neq 0$):

$$\partial_t a + \lambda_2 \partial_x a + \frac{\partial \lambda_2}{\partial R^1} a^2 + \frac{a}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2} \left(\partial_t v^2 + \lambda_2 \partial_x v^2 \right) = 0.$$

Now fix $x_0 \in \mathbb{R}$, consider the curve $(t, x_1(t))$ where

$$\frac{d x_1(t)}{dt} = \lambda_2(u(t, x_1(t))),$$

$$x_1(0) = x_0.$$

Then v^1 is constant along $(t, x_1(t))$ since

$$\frac{d}{dt}(v^1(t, x_1(t))) = \partial_t v^1(t, x_1(t)) + \lambda_2(u(t, x_1(t)))\, \partial_x v^1(t, x_1(t)) = 0.$$

Hence

$$v^1(t, x_1(t)) = v^1(0, x_0) =: v^1_0.$$

Next, set

$$\mathcal{J}(t) := e^{\int_0^t \left(\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2}\left(\partial_t v^2 + \lambda_2 \partial_x v^2\right)\right)(\tau, x_1(\tau))\, d\tau},$$

$$\rho(t) := a(t, x_1(t)) = \partial_x v^1(t, x_1(t)).$$

We will transform the evolution equation for a that we derived above into an evolution equation for $\mathcal{J}$ and ρ. Since v^1 is constant along $(t, x_1(t))$, we have that, as a function of v,

$$\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2}$$

depends only on v^2 along this curve. Therefore, setting

$$r(s) := \int_0^s \frac{1}{\lambda_1 - \lambda_2} \frac{\partial \lambda_2}{\partial R^2}(v_0', w)\, dw,$$

we have

$$\frac{d}{ds} r(s) = \frac{1}{\lambda_1 - \lambda_2} \frac{\partial \lambda_2}{\partial R^2}$$

and thus

$$\zeta(t) = e^{\int_0^t \left(\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2}\left(\partial_t v^2 + \lambda_2 \partial_x v^2\right)\right)(\tau, x_1(\tau))\, d\tau}$$

$$= e^{\int_0^t \frac{d}{d\tau} r(v^2(\tau, x_1(t)))\, d\tau}$$

$$= e^{r(v^2(t, x_1(t))) - r(v^2(0, x_1(0)))}$$

$$= e^{r(v^2(t, x_1(t))) - r(v^2(0, x_0))}.$$

Compute

$$\frac{d\zeta}{dt}(t) = e^{r(v^2(t, x_1(t))) - r(v^2(0, x_0))} \frac{d}{dt}\left(r(v^2(t, x_1(t)))\right)$$

$$= \zeta(t)\left(\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2}\right)(t, x_1(t)) \frac{d}{dt}\left(v^2(t, x_1(t))\right)$$

$$= \zeta(t)\left(\frac{1}{\lambda_2 - \lambda} \frac{\partial \lambda_2}{\partial R^2} \frac{d}{dt} v^2\right)(t, x_1(t))$$

$$\frac{d\rho}{dt}(t) = \frac{d}{dt}\left(a(t, x_1(t))\right)$$

$$= \left(-\frac{\partial \lambda_2}{\partial R^1} a^2 - \frac{a}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2} \left(\partial_t v^2 + \lambda_2 \partial_x v^2 \right) \right)(t, x_1(t))$$

$$= -\left(\frac{\partial \lambda_2}{\partial R^1} a^2\right)(t, x_1(t)) - a(t, x_1(t)) \left(\frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial R^2} \frac{d}{dt} v^2 \right)(t, x_1(t))$$

Hence, since $\rho(t) = a(t, x_1(t))$, and using $\frac{d\zeta}{dt}$

$$\frac{d\rho}{dt} = -\frac{\partial \lambda_2}{\partial R^1} \rho^2 - \frac{\rho}{\zeta} \frac{d\zeta}{dt}.$$

Thus (note that $\zeta(t) \neq 0$)

$$\frac{d}{dt}\left((\zeta\rho)^{-1}\right) = -\frac{1}{(\zeta\rho)^2} \left(\rho \frac{d\zeta}{dt} + \zeta \frac{d\rho}{dt} \right)$$

$$= -\frac{1}{(\zeta\rho)^2} \left(\rho \frac{d\zeta}{dt} + \zeta \left(-\frac{\partial \lambda_2}{\partial R^1} \rho^2 - \frac{\rho}{\zeta} \frac{d\zeta}{dt} \right) \right)$$

$$= -\frac{1}{\zeta} \frac{\partial \lambda_2}{\partial R^1},$$

provided $\rho \neq 0$. Since $\rho(t) = a(t, x_1(t)) = \partial_x v^1(t, x_1(t))$ and v^1 is constant along $(t, x_1(t))$, if $\partial_x v^1 < 0$ somewhere initially, then there exists a region consisting of characteristics starting on an interval on $\{t=0\} \times \mathbb{R}$ such that $\rho \neq 0$.

Integrating

$$(\zeta(t)\beta(t))^{-1} = (\zeta(0)\beta(0))^{-1} + \int_0^t \frac{1}{\zeta(\tau)} \frac{\partial \lambda_2}{\partial r^1}(v(\tau, x_1(\tau)))\, d\tau .$$

Note that $\zeta(0) = 1$. Solving for $\beta(t)$

$$\beta(t) = \frac{1}{\zeta(t)} \frac{1}{\frac{1}{\beta(0)} + \int_0^t \frac{1}{\zeta(t)} \frac{\partial \lambda_2}{\partial r^1}(v(\tau, x_1(\tau)))\, d\tau}$$

$$= \frac{\beta(0)}{\zeta(t)} \frac{1}{1 + \beta(0)\int_0^t \frac{1}{\zeta(t)} \frac{\partial \lambda_2}{\partial r^1}(v(\tau, x_1(\tau)))\, d\tau} .$$

Changing r_i by $-r_i$ if needed we can assume that $\frac{\partial \lambda_2}{\partial r^1} > 0$ (recall that $\frac{\partial \lambda_i}{\partial r^j}$ is proportional to $\nabla \lambda_i \cdot r_i \neq 0$, $i \neq j$). From the equations for v, we see (integrating along the characteristics), we have that v remains bounded, thus does ζ. Therefore, the only way $\beta = \partial_x v^1$ could exist for all times is if $\beta(0)$ is always > 0. A similar calculation with v^2 finishes the proof.

$\square$

Remark. Notice that the theorem does not quite reveal the mechanism of blow-up, i.e., it says that some x-derivative has to become infinite but does not quite say why. For Burgers' equation, we saw that the mechanism is the intersection of the characteristics.

Non-uniqueness of weak solutions

Let us return to the example of solution to the Riemann problem for Burgers' equation with data

$$h(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

Recall that we found that

$$u(t,x) = \begin{cases} 0, & x < 0, \\ \frac{x}{t}, & 0 < x < t, \\ 1, & x > t, \end{cases}$$

was a weak solution. However, one can verify that

$$u(t,x) = \begin{cases} 0, & x < t/2 \\ 1, & x > t/2 \end{cases}$$

is also a weak solution. This illustrates an important fact about systems of conservation laws: in general, weak solutions are not unique.

Entropy solutions

The non-uniqueness of weak solutions is possibly caused because our definition of weak solutions is so general that it possibly includes some "non-physical" solutions. Is there a way of restricting our definition of weak solutions so that we obtain a unique "physical" solution? The answer is yes.

Def. Consider a scalar conservation law

$$\partial_t u + \partial_x (F(u)) = 0.$$

A weak solution is called an entropy solution if

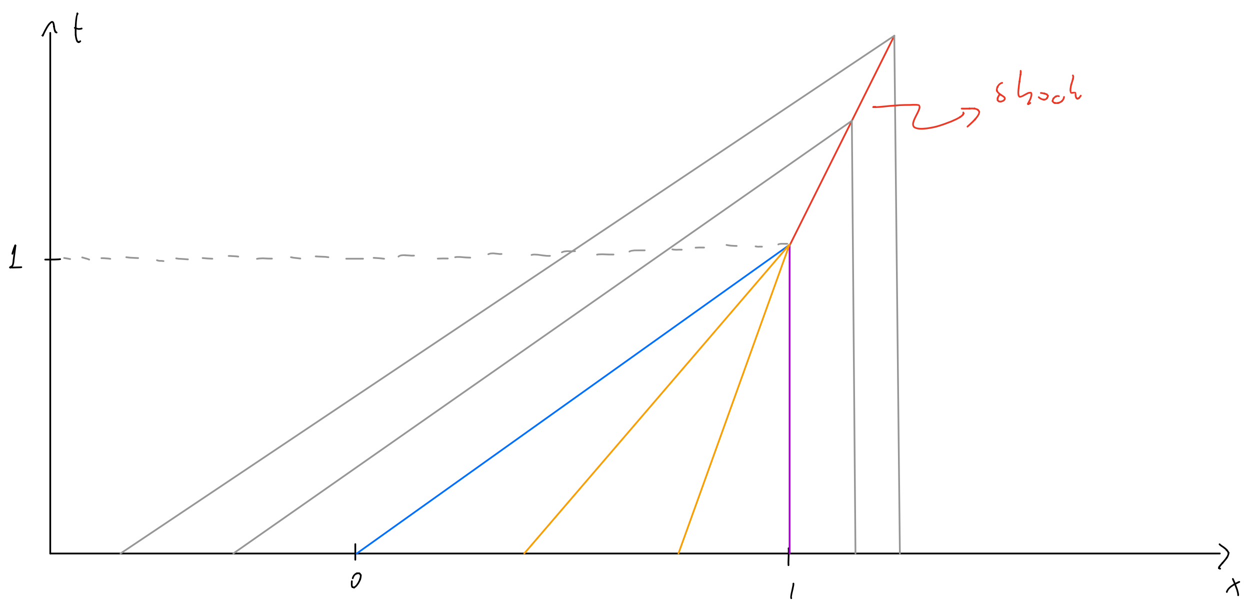
$$F'(u_l) > \sigma > F'(u_r)$$

along any shock curve, where we recall that $\sigma = \dot{r}$.

The inequality is known as the entropy condition.

Remark. Entropy solutions can also be defined for systems of conservation laws.

The idea of this definition is the following. As we have seen, we can have the formation of shocks due to the intersection of characteristics, i.e., we encounter discontinuities in the solution due to the crossing of characteristics when we move forward in time. However, we can hope that if we start at some point and move backwards in time along a characteristics, we do not cross any other. This is illustrated in the following example we saw of shock formation for Burgers' equation:



For $\partial_t u + \partial_x (F(u)) = \partial_t u + F'(u) \partial_x u = 0$ the characteristics are $(t, F'(h(\alpha))t + \alpha)$, where $h(x) = u(0,x)$. (The solution is constant along the characteristics.) The desired situation will happen if when the characteristics meet the one on the left is "faster" than the one on the right, i.e.,

$$F'(h(\alpha_l)) > F'(h(\alpha_r)),$$

or since u is constant along the characteristics and the speed of the shock curve shall be an intermediate value,

$$F'(u_l) > \sigma > F'(u_r).$$

One of the landmark results in systems of conservation laws is that, under some very general assumptions, entropy solutions are unique and exist for all time.

Final remarks

We finish this course with the following important observation. We developed some of the basic elements of PDE theory, but we barely scratched the surface of the topic of PDEs. Because this was an introductory course, we exploit at length techniques that rely on explicit formulas and on ODE arguments. This should not give readers the wrong impression that these techniques are appropriate for the study of more advanced topics in PDE. Going deeper into the topic requires developing new tools (often connected to functional analysis and geometry) that are very different of the ones we employed in this course.