HOMEWORK 8

MATH 3120

Unless stated otherwise, the notation below is as in class. You can assume that all functions are C^∞ unless explicitly assumed otherwise.

Question 1. In this question, you will provide a blow-up proof for Burgers' equation different than the one given in class.

(a) Differentiate the equation and show that the variable $\psi := \partial_x u$ satisfies the equation

$$\partial_t \psi + u \partial_x \psi = -\psi^2. \tag{1}$$

(b) Show that (1) implies that $y(t) := \psi(t, x(t, \alpha))$ satisfies the Riccati equation $\dot{y} = -y^2$ along the characteristics $(t, x(t, \alpha))$.

(c) Use your knowledge of ODE to conclude, from the Riccati equation, blow-up for Burgers.

Question 2. Define

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} := \sup_{x \in \mathbb{R}} |u(t, x)|.$$

Write Burgers' equation as an ODE along the characteristics (similarly to what you did for ψ in the previous problem) to conclude that

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|u(0, \cdot)\|_{L^\infty(\mathbb{R})} = \|h\|_{L^\infty(\mathbb{R})},$$

i.e., the L^∞ norm is conserved over time.

Question 3. Prove that if u is a weak solution that is C^∞ then it is in fact a classical solution.

Question 4. Consider the Cauchy problem for Burgers' equation with data given by

$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

(a) Show that the solution is given by

$$u(t, x) = \begin{cases} 1, & x \leq t, t < 1, \\ \frac{1-x}{1-t}, & t < x < 1, t < 1, \\ 0, & x \geq 1, t < 1. \end{cases}$$

(b) The denominator of $\frac{1-x}{1-t}$ approaches zero when $t \to 1^-$. Does that mean that $|u(t, x)| \to \infty$ as $t \to 1^-$? Does this not contradict your result from question 2? What exactly is becoming singular when the characteristics intersect at $(1, 1)$?

(c) Let $0 < \beta < 1$ and define, for $t \geq 1$

$$\tilde{u}(t, x) = \begin{cases} 1, & x < \beta t + 1 - \beta, \\ 0 & x > \beta t + 1 - \beta. \end{cases}$$

Show that v given by

$$v(t,x) = \begin{cases} u(t,x), & 0 \leq t < 1, \\ \tilde{u}(t,x), & t \geq 1, \end{cases}$$

is a weak solution if and only if $\beta = 1/2$. (This was essentially done in class. Here, you have to work out the calculations in more detail, including the case when Ω might intersect the region $\{t \leq 1\}$.)