HOMEWORK 1

MATH 3120

The notation and terminology below is the same used in class.

Question 1. Review multivariable calculus, especially the chain rule in several variables and vector identities/operators.

Question 2. Verify whether the given function is a solution of the given PDE:

(a) $u(x,y) = y\cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x,y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Question 3. For each PDE seen as example in the first class (Laplace's equation, heat equation, wave equation, Schödinger's equation, Maxwell's equation, Euler and Navier-Stokes equations), state whether it is a scalar PDE (i.e., single PDE) or a system of PDEs, its order, and whether it is a linear or non-linear PDE. (We have not yet defined what linear vs. non-linear PDEs means, but your knowledge of ODEs should suffice for this homework set.)

Question 4. Consider a linear homogeneous PDE. Explain why any linear combination of solutions is also a solution. (Again, use your knowledge of ODE to define linearity here.)

Question 5. Consider Maxwell's equations:

$$\begin{aligned}
\operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\
\operatorname{div} B &= 0, \\
\frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\
\frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J.
\end{aligned}$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?

Question 6. Consider Euler's equations:

$$\partial_t \varrho + u^i \partial_i \varrho + \varrho \partial_i u^i = 0,$$

$$\varrho(\partial_t u^j + u^i \partial_i u^j) + \nabla^j p = 0,$$

where we recall that $p = p(\varrho)$. A fluid is called *incompressible* if $\varrho = constant$, in which case we can set $\varrho = 1$. In this case, the equations describing the fluid motion are

$$\begin{aligned} \partial_t u^j + u^i \partial_i u^j + \nabla^j p &= 0, \\ \partial_i u^i &= 0, \end{aligned}$$

which are called the *incompressible Euler equations.* For an incompressible fluid, however, the pressure is no longer given by $p = p(\varrho)$, since the pressure would then be constant, but experiments show that the pressure can vary even if the density remains (approximately) constant. Show that in the case of the incompressible Euler equations, the pressure is given as a solution to

$$\Delta p = -\partial_j u^i \partial_i u^j.$$

Question 7. Consider the incompressible Euler equations (see previous question):

$$\begin{aligned} \partial_t u^j + u^i \partial_i u^j + \nabla^j p &= 0, \\ \partial_i u^i &= 0. \end{aligned}$$

The *vorticity* ω of the fluid is defined as

$$\omega := \operatorname{curl} u.$$

The vorticity is an important physical quantity; it measures, as the name suggests, "eddies" in the fluid. It is, therefore, important to know how it changes in time and space (i.e., what the dynamics of the vorticity is). Show that ω satisfies the following PDE:

$$\partial_t \omega + \nabla_u \omega - \nabla_\omega u = 0.$$

Above, the operators ∇_u and ∇_ω are defined as follows. For any vector field X, ∇_X is a short hand notation for $X \cdot \nabla$, i.e.,

$$\nabla_X := X \cdot \nabla,$$

where we recall that $X \cdot \nabla$ has been defined in class as

$$X \cdot \nabla = X^i \partial_i.$$