VANDERBILT UNIVERSITY

MATH 3120 – INTRO DO PDES

Test 2

NAME: Solutions

Directions. This exam contains four questions. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc).

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	**Points**
1 (25 pts)	
2 (25 pts)	
3 (25 pts)	
4 (25 pts)	
TOTAL (100 pts)	

Question 1. (25 pts) Answer the questions below. Justify your answers.

(a) What is the method of separation of variables? Is it guaranteed to always produce a solution to a PDE?

(b) What is the difference between a formal solution and an actual solution to a PDE?

(c) Can a formal solution also be a classical solution? Can it be a generalized solution?

(d) Let f be a function defined on $(-L, L)$, $L > 0$, and $F.S.\{f\}$ its Fourier series. Is it true that for any $x \in (-L, L)$ we have that $f(x) = F.S.\{f\}(x)$?

(e) Let $f : (-2, 2) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & -2 < x \leq -1, \\ 2, & -1 < x < 0, \\ 1, & 0 \leq x \leq 1 \\ x, & 1 < x < 2. \end{cases}$$

Let $F.S.\{f\}$ be its Fourier series. Find $F.S.\{f\}(-1)$, $F.S.\{f\}(0)$, $F.S.\{f\}(1)$, and $F.S.\{f\}(1.5)$, i.e., the values of the Fourier series of f at the points $x = -1, 0, 1, 1.5$. *Hint:* you do not need to compute the Fourier series of f to solve this problem.

Solution 1. (a) The method of separation of variables consists in attempting to solve a PDE by supposing that the unknown function is a product of functions of single variables, each of which depends on one of the independent variables of the problem. Thus, if the unknown is $u = u(x_1, \ldots, x_n)$, one tries a solution of the form $u(x_1, \ldots, x_n) = X_1(x_1) \cdots X_n(x_n)$. The method is not guaranteed to produce a solution.

(b) A formal solution is an expression that provides a candidate for a solution. It typically consists of a formula involving a series, with no further information on the convergence of the series or other information that makes the expression mathematically well-defined.

(c) Yes in both cases. If a formal solution given as a series converges to a C^k function (respectively piece-wise C^k), with k greater or equal to the order of the equation, then the formal solution yields a classical (respectively generalized) solution. The type of convergence involved can vary from problem to problem.

For parts (d) and (e), we recall the following theorem.

Theorem 1. *Let f be a piecewise C^1 function defined on $[-L, L]$. Then, for any $x \in (-L, L)$,*

$$F.S.\{f\}(x) = \frac{1}{2}(f(x^+) + f(x^-)).$$

For $x = \pm L$, the series converges to $\frac{1}{2}(f(-L^+) + f(L^-))$.

(d) No. Take the function

$$f(x) = \begin{cases} -1, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

By the above theorem, $F.S.\{f\}(0) = 0$, but $f(0) = -1$.

(e) Using the above theorem, $F.S.\{f\}(-1) = (-1 + 2)/2 = 1/2$, $F.S.\{f\}(0) = (2 + 1)/2 = 3/2$, $F.S.\{f\}(1) = (1 + 1)/2 = 1$, and $F.S.\{f\}(1.5) = 1.5$.

Question 2. (25 pts) Consider the following initial-boundary value problem for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0 \qquad \text{in } (0, L) \times (0, \infty), \tag{1a}$$

$$u(x, 0) = f(x) \quad 0 \le x \le L, \tag{1b}$$

$$u_t(x, 0) = g(x) \quad 0 \le x \le L, \tag{1c}$$

$$u(0, t) = 0 \qquad t \ge 0, \tag{1d}$$

$$u(L, t) = 0 \qquad t \ge 0. \tag{1e}$$

(a) What compatibility conditions do f and g have to satisfy?

(b) Using separation of variables, write two ordinary differential equations that are consequence of equation (1a).

(c) Find a formal solution to the initial-boundary value problem (1).

(d) State sufficient conditions on f and g that guarantee that the formal solution you found in (c) is an actual solution to the problem.

(e) Explain how a formal solution to (1) can be showed to be an actual solution under the conditions you stated in (d). You are not required to provide a formal proof. Rather, outline the argument and its main steps. In doing so, state any relevant theorems you need to invoke.

Solution 2. (a) In order to have u well-defined, we need that $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$.

(b) Set $u(x, t) = X(x)T(t)$ and plug into (1a) to find

$$\frac{X''}{X} = \frac{T''}{c^2 T}.$$

Since the left-hand side depends only on x and the right-hand side only on t, both sides need to be equal to a constant λ. Thus

$$X'' = \lambda X, \text{ and } T'' = \lambda c^2 T.$$

(c) This was done in class (see class notes from February 6 and and February 14). We find

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \frac{cn\pi t}{L} + b_n \sin \frac{cn\pi t}{L}) \sin \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \text{ and } b_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

(d) and (e) We answer parts (d) and (e) together. We make an odd extension of f and g to $2L$-periodic functions $\widetilde{f}$ and $\widetilde{g}$ on $\mathbb{R}$. It follows that $\widetilde{f}(-L) = \widetilde{g}(-L) = \widetilde{f}(L) = \widetilde{g}(L) = 0$. Using D'Alembert's formula, we write a solution $\widetilde{u}$ for the wave equation on the real line with initial conditions $\widetilde{f}$ and $\widetilde{g}$. Next, we consider the Fourier series of $\widetilde{u}$, which amounts to consider the Fourier series of $\widetilde{f}$ and $\widetilde{g}$. Because $\widetilde{f}$ is odd, the coefficients a_n and b_n of $F.S.\{\widetilde{u}\}$ agree with the expressions for a_n and b_n in part (c). With trigonometric identities for the sine and cosine of the sum of angles, we expand D'Alemberts formula for $\widetilde{u}$, and observe that the resulting expression agrees with the formal solution u found in (c), and also satisfies the boundary conditions. Therefore, the formal solution in (c) will be an actual solution provided that we can apply theorems for convergence of Fourier series and its derivatives. A theorem for convergence of the Fourier series was stated above, and a theorem for differentiation of Fourier series is the following.

Theorem 2. *Let f be continuous on $[-L, L]$. Suppose that $f(-L) = f(L)$, and that f is piecewise C^2. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. I.e., if*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}),$$

then

$$f'(x) = \sum_{n=1}^{\infty}(a_n(\cos\frac{n\pi x}{L})' + b_n(\sin\frac{n\pi x}{L})'),$$

whenever $f'(x)$ equals its Fourier series. Equivalently,

$$f'(x) = \sum_{n=1}^{\infty}(-a_n\frac{n\pi}{L}\sin\frac{n\pi x}{L} + b_n\frac{n\pi}{L}\cos\frac{n\pi x}{L}).$$

One simple condition guaranteeing the convergence of $F.S.\{\widetilde{u}\}$ and its derivatives on $(0, L)$ is that f and g be smooth.

See the class notes of February 21 for a detailed presentation of the above argument.

Question 3. (25 pts) Consider the following initial-boundary value problem for the heat equation:

$$u_t - ku_{xx} = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{2a}$$
$$u(x, 0) = f(x) \quad 0 \leq x \leq L, \tag{2b}$$
$$u(0, t) = 0 \quad t \geq 0, \tag{2c}$$
$$u(L, t) = 0 \quad t \geq 0. \tag{2d}$$

(a) The following expression is a formal solution to problem (2) (you do not need to establish this):

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}, \tag{3}$$

where

$$b_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx.$$

What happens to the formal solution when $t \to \infty$? How do you interpret this result?

(b) Determine the formal solution (3) when $k = 1$, $L = \pi$, and

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{\pi}{2}, \\ 2, & \frac{\pi}{2} < x \leq \pi. \end{cases}$$

(c) Prove that for any fixed $t > 0$, the formal solution you found in (b) converges for any $x \in [0, \pi]$.

Solution 3. (a) When $t \to \infty$, the formal expression gives that $u(x, t) \to 0$, meaning that the temperature of the rod goes to zero (recall the physical interpretation of the heat equations discussed at the beginning of the course). This makes sense in light of the boundary conditions: we are considering the case where the rod's endpoints are kept at zero temperature and no other heat exchange with the environment is allowed. Hence, the rod's temperature will eventually becomes zero.

(b) We find

$$b_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx = \frac{4}{\pi} \int_{\frac{\pi}{2}}^{\pi} \sin(nx)\, dx = \frac{4}{n\pi}((-1)^{n+1} + \cos\frac{n\pi}{2}),$$

and the solution is (3) with this expression for b_n, $L = \pi$, and $k = 1$.

(c) Since the exponential increases faster than any polynomial, we have that, if $t > 0$ is fixed, then

$$\left| \frac{4}{n\pi}((-1)^{n+1} + \cos\frac{n\pi}{2})e^{-n^2 t} \right| \leq \frac{C}{n^2},$$

for some constant $C > 0$ and for all n sufficiently large. Since the series of $\frac{1}{n^2}$ converges, so does the formal solution.

Question 4. (25 pts) This question deals with the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta\Psi + V\Psi, \tag{4}$$

that was studied in class. Below, several results about the Schrödinger equation that have been established in class are recalled before stating the questions. You do not need to establish such results, only use them to answer the questions you are asked.

Assume that V is the Coulomb potential given by

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r},$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Recall that solutions to (4) need to satisfy

$$\int_{\mathbb{R}^3} |\Psi(x,t)|^2 \, dx = 1,$$

in order to be physically acceptable.

In class, we employed separation of variables to write

$$\Psi(x,t) = T(t)\Theta(\theta)\Phi(\varphi)R(r),$$

where (r,θ,φ) are spherical coordinates, with $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$.

(a) What is the correct boundary condition for Θ?

(b) In order to find Φ, in class we made the following change of variables

$$v = \cos\varphi, \tag{5}$$

and obtained a power series solution

$$P(v) = \sum_{k=0}^{\infty} a_k v^k, \tag{6}$$

where the coefficients a_k satisfy the following recurrence relation:

$$a_{k+2} = \frac{k(k+1)-\lambda}{(k+1)(k+2)} a_k, \ k = 0, 1, 2, \ldots, \tag{7}$$

where λ is a constant that comes from the separation of variables.

Explain how (5), (6), and (7) are used to determine that

$$\lambda = \ell(\ell+1),$$

for $\ell = 0, 1, 2, \ldots$.

(c) The function R satisfies the following equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left(E - V(r)\right)R = \ell(\ell+1)\frac{R}{r^2}, \tag{8}$$

where E is a constant that comes from the separation of variables.

Explain how (8) is used to show that $E < 0$.

Solution 4. (a) Since the spherical coordinates (r,θ,φ) and $(r,\theta+2\pi,\varphi)$ represent the same point in space, we must have $\Theta(\theta) = \Theta(\theta+2\pi)$.

(b) Since points with coordinates $\varphi = 0$ and $\varphi = \pi$ must be included, we need to consider, in view of (5), the solution (6) when $v = \pm 1$:

$$P(\pm 1) = \pm \sum_{k=0}^{\infty} a_k. \tag{9}$$

From (7) we have

$$a_{k+2} = \frac{k^2 + O(k)}{k^2 + O(k)} a_k = \frac{k^2 + O(k)}{k^2 + O(k)} \frac{k^2 + O(k)}{k^2 + O(k)} a_{k-2} = \cdots = \begin{cases} \frac{k^{k+2} + O(k^{k+1})}{k^{k+2} + O(k^{k+1})} a_0, & k \text{ even}, \\ \frac{k^{k+1} + O(k^{k})}{k^{k+1} + O(k^{k})} a_1, & k \text{ odd}. \end{cases}$$

It follows that

$$\lim_{k \to \infty} a_k \neq 0,$$

and therefore (9) diverges by the divergence test, *unless* (6) is in fact a *finite sum*; i.e., *unless* $a_k = 0$ *for all* k *greater than a certain* ℓ. Hence, we must have, form some non-negative integer ℓ,

$$a_{\ell+2} = 0 = \frac{\ell(\ell+1) - \lambda}{(\ell+1)(\ell+2)} a_\ell,$$

which implies

$$\lambda = \ell(\ell+1),$$

provided that $a_\ell \neq 0$.

(c) First we show that E must be real. To see this, multiply equation (8) by $r^2 R^*$, where R^* is the complex conjugate of R, and integrate from 0 to ∞:

$$\int_0^\infty R^* \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) dr - \frac{2\mu}{\hbar^2} \int_0^\infty V|R|^2 r^2\, dr - \ell(\ell+1) \int_0^\infty |R|^2\, dr = -\frac{2\mu}{\hbar^2} E \int_0^\infty |R|^2 r^2\, dr, \tag{10}$$

where we used that $|R|^2 = R^* R$. Integrating by parts the first term,

$$\int_0^\infty R^* \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) dr = -\int_0^\infty \frac{dR^*}{dr} \frac{dR}{dr} r^2\, dr + R^* r^2 \frac{dR}{dr}\Big|_0^\infty = -\int_0^\infty \frac{dR^*}{dr} \frac{dR}{dr} r^2\, dr \tag{11}$$

where it has been assumed that R^* and $\frac{dR}{dr}$ vanish sufficiently fast at ∞. Writing

$$R = R_R + i R_C,$$

where R_R and R_C are real-valued, it comes

$$\frac{dR^*}{dr}\frac{dR}{dr} = (\frac{dR_R}{dr} - i\frac{dR_C}{dr})(\frac{dR_R}{dr} + i\frac{dR_C}{dr}) = \left(\frac{dR_R}{dr}\right)^2 + \left(\frac{dR_C}{dr}\right)^2,$$

and we conclude that $\frac{dR^*}{dr}\frac{dR}{dr}$ is real-valued. But from (10) and (11) we have

$$E = \frac{\int_0^\infty \frac{dR^*}{dr}\frac{dR}{dr} r^2\, dr + \frac{2\mu}{\hbar^2}\int_0^\infty V|R|^2 r^2\, dr + \ell(\ell+1)\int_0^\infty |R|^2\, dr}{\frac{2\mu}{\hbar^2}\int_0^\infty |R|^2 r^2\, dr}. \tag{12}$$

Therefore, since all terms on the right-hand side are real, we conclude that E is real as well.

Next, we show that $E < 0$. Let us investigate the behavior of (8) for large values of r, i.e., $r \gg 1$. Then we can neglect the terms that contain $\frac{1}{r}$ and (8) gives, after expanding the terms in $\frac{d}{dr}$,

$$\frac{d^2 R}{dr^2} \approx -\frac{2\mu E}{\hbar^2} R. \tag{13}$$

But for $r \gg 1$ we also have the approximation

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} \approx r\frac{d^2R}{dr^2},$$

so that

$$\frac{d^2(rR)}{dr^2} = r\frac{d^2R}{dr^2} + 2\frac{dR}{dr} \approx r\frac{d^2R}{dr^2}. \tag{14}$$

Hence, multiplying (13) by r and using (14),

$$\frac{d^2(rR)}{dr^2} \approx -\frac{2\mu E}{\hbar^2}(rR).$$

This approximate equation can be easily solved, producing

$$rR \approx e^{\pm\frac{\sqrt{-2\mu E}}{\hbar}r}.$$

If $E \geq 0$, then R is a complex function which satisfies

$$|rR| \approx 1 \text{ for } r \gg 1.$$

Then the integral

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2 \, dx = \left(\int_0^{2\pi}\int_0^{\pi} |Y(\phi,\theta)|^2 \sin\phi \, d\phi d\theta\right)\left(\int_0^{\infty} |R(r)|^2 r^2 \, dr\right)$$

diverges since $|R(r)|^2r^2 \approx 1$ for large r. Consequently, condition

$$\int_{\mathbb{R}^3} |\Psi(x,t)|^2 \, dx = 1,$$

fails, and this does not produce a physically sensible solution. Therefore, we must have $E < 0$.