VANDERBILT UNIVERSITY

MATH 3120 – INTRO DO PDES

Practice problems for HW 5

Prove the following properties of the Fourier transform. In doing so, you can use the following property

$$(\hat{f})\check{} = f, \tag{1}$$

which you need not to prove. The properties below (as well as (1)) have been stated in class. You are referred to the class notes for further details on the notation.

Question 1. The Fourier transform and its inverse are linear.

Question 2.

$$\left(\frac{\partial f}{\partial x_j}\right)\hat{} = ik_j \hat{f},$$

and

$$\left(\frac{\partial^{m_1+\cdots+m_n} f}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\right)\hat{} = (ik_1)^{m_1}\cdots(ik_n)^{m_n}\hat{f}.$$

Hint: integration by parts.

Question 3. $\overline{\hat{f}(k)} = \hat{f}(-k)$ if f is a real function.
Hint: change of variables.

Question 4. $\mathcal{F}(f * g) = (2\pi)^n \hat{f}\hat{g}$.
Hint: change of variables and change in the order of integration.

For questions 5 to 7, you can assume that $n = 1$.

Question 5. $\mathcal{F}(f(x-a)) = e^{-iak}\hat{f}(k)$.

Question 6. $\mathcal{F}(e^{iax}f(x)) = \hat{f}(k-a)$.

Question 7. $\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}(\frac{k}{a})$, $a \neq 0$.
Hint: change of variables.

Question 8. Show that the Fourier transform of $e^{-a|x|}$, $a > 0$, $x \in \mathbb{R}$, is $\frac{2a}{a^2+k^2}$.

Question 9. (a) Recall that the "δ-function" is characterized by

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\,dx = f(0), \tag{2}$$

for "any" function $f : \mathbb{R} \to \mathbb{R}$. Find the Fourier transform of δ.

(b) Next, recall the Heaviside function:

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Find $\widehat{H}$.

Remark: If you have not learned about the delta-function and the Heaviside function, you may want to skip this problem. The use of quotation marks above is because the delta function is not really a function, and equality (2) does not hold for arbitrary functions, although it does hold for a large class of functions that appear in applications.

Question 10. In this question, we explore a connection between the Fourier series and the Fourier transform. Since our goal here is primarily to motivate this connection rather than establishing rigorous results, you can assume that the manipulations indicated below are valid and that f equals its Fourier series.

(a) Consider the Fourier series for a function f defined on $[-L, L]$. Show that it can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}},$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} \, dx.$$

Hint: Euler's formula.

(b) Set $k = \frac{n\pi}{L}$ to obtain:

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^{L} f(y) e^{-iky} \, dy \right) e^{ikx} \frac{\pi}{L}.$$

(c) Show that taking the limit $L \to \infty$, we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky} \, dy \right] e^{ikx} \, dk.$$

The term between brackets is the Fourier transform of f, and the above equality is (1).
Hint: notice that the distance between two successive k's is $\Delta k = \frac{\pi}{L}$, and think of the definition of an integral as the limit of sums.