VANDERBILT UNIVERSITY

MATH 3120 – INTRO DO PDES

Practice problems on Green's functions and the wave-equation

In the questions below, we follow the notation employed in class.

Question 1. Verify that the formulas given in class for $\mathbb{R}^n_+$ and $B_1(0)$ are in fact the formulas for the Green function of these regions.

Question 2. Here you have to complete the argument, outlined in class, showing that solutions to the wave equation in $\mathbb{R}^2$ are obtained from solutions to the wave equation in $\mathbb{R}^3$. Let u solve

$$\begin{aligned} u_{tt} - \Delta u = 0 \quad & \text{in } \mathbb{R}^2 \times (0, \infty), \\ u = g \quad & \text{on } \mathbb{R}^2 \times \{t = 0\}, \\ u_t = h \quad & \text{on } \mathbb{R}^2 \times \{t = 0\}. \end{aligned}$$

(a) Define $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$, $\bar{g}(x_1, x_2, x_3) = g(x_1, x_2)$, and $\bar{h}(x_1, x_2, x_3) = h(x_1, x_2)$. Show that $\bar{u}$ solves

$$\begin{aligned} \bar{u}_{tt} - \Delta \bar{u} = 0 \quad & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \bar{u} = \bar{g} \quad & \text{on } \mathbb{R}^3 \times \{t = 0\}, \\ \bar{u}_t = \bar{h} \quad & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{aligned}$$

(b) Denote $x = (x_1, x_2)$ and $\bar{x} = (x_1, x_2, 0)$. Using Kirchhoff's formula in $n = 3$ (derived in class), show that

$$u(x,t) = \partial_t \left(t \frac{1}{3\alpha(3)t^2} \int_{\partial B_t(\bar{x})} \bar{g} \right) + \frac{1}{3\alpha(3)t^2} \int_{\partial B_t(\bar{x})} \bar{h}.$$

(c) Letting $\gamma(y) = \sqrt{t^2 - |y - x|^2}$, show that

$$\frac{1}{3\alpha(3)t^2} \int_{\partial B_t(\bar{x})} \bar{g} = \frac{2}{4\pi t^2} \int_{B_t(x)} g(y) \sqrt{1 + |\nabla \gamma(y)|^2}\, dy,$$

(d) Show that $\sqrt{1 + |\nabla \gamma(y)|^2} = \frac{t}{\sqrt{t^2 - |y-x|^2}}$, and that

$$\frac{1}{3\alpha(3)t^2} \int_{\partial B_t(\bar{x})} \bar{g} = \frac{t}{2} \frac{1}{\pi t^2} \int_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}}\, dy.$$

(e) Use

$$t^2 \frac{1}{\pi t^2} \int_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}}\, dy = t \frac{1}{\pi} \int_{B_1(0)} \frac{g(x + tz)}{\sqrt{1 - |z|^2}}\, dz$$

to conclude that

$$u(x,t) = \frac{1}{2} \frac{1}{\pi t^2} \int_{B_t(x)} \frac{t g(y) + t^2 h(y) + t \nabla g(y) \cdot (y - x)}{\sqrt{t^2 - |y - x|^2}}\, dy.$$