

Initial-boundary value problem for the 1D-wave equation: the method of separation of variables.

We now consider the problem

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } (0, L) \times (0, \infty)$$

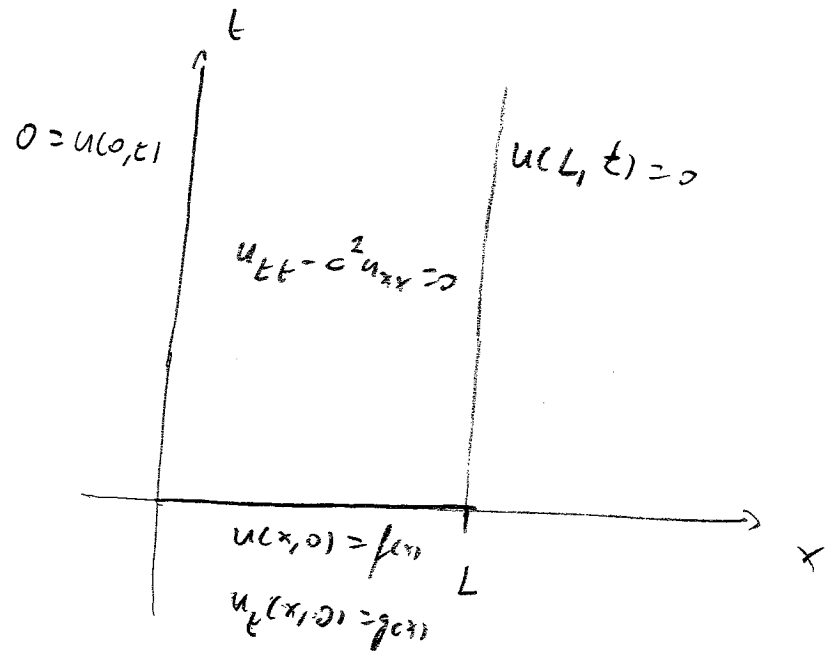
$$\text{B.C. } \begin{cases} u(0, t) = u(L, t) = 0 \end{cases}, \quad t \geq 0$$

$$\text{I.C. } \begin{cases} u(x, 0) = f(x), & 0 \leq x \leq L \\ u_t(x, 0) = g(x), & 0 \leq x \leq L \end{cases}$$

where $L > 0$, f and g are given functions.

Notice that f and g must satisfy the compatibility conditions:

$$f(0) = f(L) = 0, \quad g(0) = g(L) = 0$$



In order to solve the problem, we are going to seek solutions that are functions of x and t that are products of a function only of x and a function only of T . I.e., we suppose that

$$u(x, t) = \bar{X}(x) T(t)$$

and see if this leads us to a solution. Plugging into the equation:

$$(\bar{X} T)_{tt} - c^2 (\bar{X} T)_{xx} = 0$$

$$\bar{X} T'' - c^2 T \bar{X}'' = 0 \quad \text{or} \quad \text{yet} \quad \frac{\bar{X}''}{\bar{X}} = \frac{T''}{c^2 T} \quad \text{provided that}$$

\bar{X} or T are not zero. But, $\frac{\bar{X}''}{\bar{X}}$ is a function only of x whereas

$\frac{T''}{c^2 T}$ is a function only of t . Thus, the only way that equality

can hold is if both $\frac{\bar{X}''}{\bar{X}}$ and $\frac{T''}{c^2 T}$ are equal to a

constant, which we denote λ .

Thus: $\frac{\bar{X}''}{\bar{X}} = \lambda$ and $\frac{T''}{c^2 T} = \lambda$. These are now two

ODEs for \bar{X} and T respectively.

Analysis of \bar{X} The equation reads $\bar{X}'' - \lambda \bar{X} = 0$. The characteristic equation is $r^2 - \lambda = 0$, $r = \pm\sqrt{\lambda}$, yielding the general solution

$$\bar{X}(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \quad \text{if } \lambda > 0$$

$$\bar{X}(x) = c_1 \cos(\sqrt{-\lambda}x) + c_2 \sin(\sqrt{-\lambda}x) \quad \text{if } \lambda < 0$$

$$\bar{X}(x) = c_1 + c_2 x \quad \text{if } \lambda = 0$$

Next, recall that $u(0,t) = 0 = \bar{X}(0)T(t)$ and $u(L,t) = 0 = \bar{X}(L)T(t)$, implying $\bar{X}(0) = \bar{X}(L) = 0$ since $T \neq 0$.

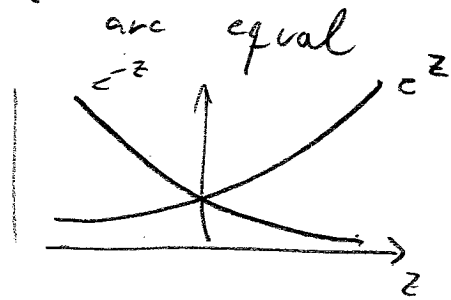
Let's investigate the three cases above. (Note that we do not know the value of λ).

If $\lambda > 0$, then we have

$$\bar{X}(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \text{ and } \bar{X}(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L} = 0$$

$$\Rightarrow \bar{X}(L) = c_1 (e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) = 0. \text{ Since } e^z \text{ and } e^{-z} \text{ are equal}$$

only when $z=0$, and $L \neq 0$, $\lambda \neq 0$, this implies $c_1 = 0$



But then $c_2 = 0$ and $\bar{X}(x) = 0$, giving $u(x,t) = \bar{X}(x)T(t) = 0$. This cannot yield a solution except in the (uninteresting) case $f=g=0$. Thus we cannot have $\lambda > 0$.

If $\lambda = 0$, then $\bar{X}(0) = c_1 = 0$ and $\bar{X}(L) = c_1 + c_2 L = 0$, which again gives $u(x,t) = 0$, so we cannot have $\lambda = 0$.

If $\lambda < 0$, then we write $-\lambda = \mu$, with $\mu > 0$, and we can write

$$\bar{X}(x) = c_1 \cos(\sqrt{\mu}x) + c_2 \sin(\sqrt{\mu}x).$$

plugging $x=0$ and using $X(0)=0$ gives $c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$,

thus $X(x) = c_2 \sin(\sqrt{\mu} x)$. Now, plug $x=L$ to get

$$X(L) = c_2 \sin(\sqrt{\mu} L) = 0, \text{ so } c_2 = 0 \text{ or } \sin(\sqrt{\mu} L) = 0.$$

If $c_2 = 0$ then again $u(x,t) = 0$, which cannot give a solution except when $f=g=0$. Thus, we must have

$$\sin(\sqrt{\mu} L) = 0.$$

But this means that $\sqrt{\mu} L$ must be a multiple of π , i.e.,

$$\sqrt{\mu} L = n\pi, \quad n = 1, 2, 3, \dots \quad (\text{we don't include } n \leq 0 \text{ because } \mu > 0).$$

This means that $\mu = \frac{n^2 \pi^2}{L^2}$ (so $\lambda = -\frac{n^2 \pi^2}{L^2}$). Therefore, we found

what λ , or equivalently, μ has to be, except that we can assume infinitely many values according to $n = 1, 2, 3, \dots$

In other words, the problem $\bar{x}'' - \lambda \bar{x} = 0$, $\bar{x}(0) = \bar{x}(L) = 0$ admits non-trivial solutions only when $-\lambda = \mu > 0$ (i.e., $\lambda < 0$), in which case it has infinitely many solutions, one for each $n = 1, 2, 3, \dots$. So we write

$$\bar{x}_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

We can now move to the T equation, using that $-\lambda = \mu = \frac{n^2\pi^2}{L^2}$.

$$\frac{T''}{c^2 T} = \lambda \Rightarrow T_n'' + \frac{c^2 n^2 \pi^2}{L^2} T_n = 0, \text{ where we write } T_n \text{ because}$$

this is a different equation for each n . From ODE theory, we know that the solutions T_n are given by

$$T_n(t) = a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right)$$

where a_n and b_n are arbitrary constants.

Since $u = \sum T$, we see that we have obtained infinitely many functions u_n , $n = 1, 2, 3, \dots$ given by

$$u_n(x, t) = \sum_n(x) T_n(t)$$

$$u_n(x, t) = \left(a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right) \right) \sin\left(\frac{n\pi}{L}x\right)$$

for $n = 1, 2, 3, \dots$. All these u_n 's solve $u_{tt} - c^2 u_{xx} = 0$ and satisfy $u(0, t) = 0 = u(L, t)$. However, we don't know yet that they satisfy the initial condition. Imposing the initial condition will determine the coefficients a_n and b_n .