Next, let's consider the wave equation with I.C:

$$(*)\quad \begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \underset{x}{(-\infty,\infty)} \times \underset{t}{(0,\infty)} \\ u(x,0) = f(x), & -\infty < x < \infty \\ u_t(x,0) = g(x), & -\infty < x < \infty \end{cases}$$

Problem $(*)$ is called the Cauchy problem for the wave equation.

We know that solutions are written as:

$u(x,t) = F(x+ct) + G(x-ct)$, so that $u_t(x,t) = cF'(x+ct) - cG'(x-ct)$

Plugging $t=0$: $u(x,0) = F(x) + G(x) = f(x)$

$u_t(x,0) = cF'(x) - cG'(x) = g(x)$. Integrating from $0$ to $x$:

$F(x) - G(x) = \frac{1}{c}\int_0^x g(s)\,ds + C_1$. Adding with $F+G=f$:

$F(x) = \frac{1}{2c}\int_0^x g(s)\,ds + \frac{C_1}{2} + \frac{f(x)}{2}$. Plugging back into $F+G=f$

$G(x) = -\frac{1}{2c}\int_0^x g(s)\,ds - \frac{C_1}{2} + \frac{f(x)}{2}$

Replacing $x$ by $x+ct$ in $F$ and by $x-ct$ in $G$, and adding:

$$\boxed{u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\underbrace{\int_{x-ct}^{x+ct} g(s)\,ds}}\,,$$

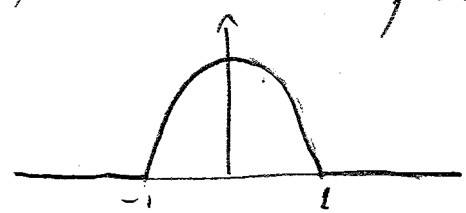
where we used that $u(x,t) = F(x+ct) + G(x-ct)$.

because $\int_0^{x+ct} - \int_0^{x-ct} = \int_0^{x+ct} + \int_{x-ct}^{0} = \int_{x-ct}^{x+ct}$

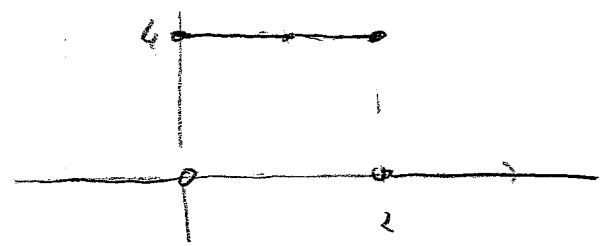
This formula is known as D'Alembert's formula.
It gives an explicit formula for the solution in terms of the initial data.

Ex: Solve the Cauchy problem for the wave equation with $c = 1$

$$f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$



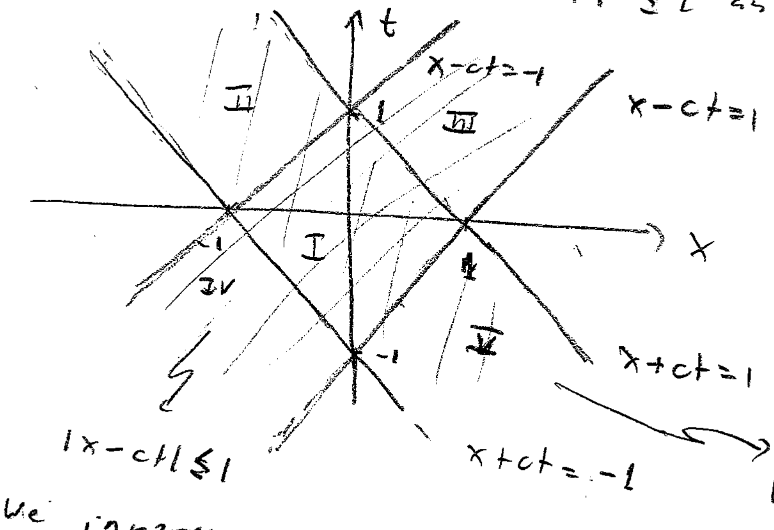

$$g(x) = \begin{cases} 4, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$



Let's use D'Alembert formula. Since the definition of $f$ changes according to $|x| \leq 1$ or $|x| > 1$, we have to consider $|x+ct| \leq 1$ and $|x+ct| > 1$. Similarly for $x-ct$.

$$|x+ct| \leq 1 \iff -1 \leq x+ct \leq 1$$

$$|x-ct| \leq 1 \iff -1 \leq x-ct \leq 1$$




There are five regions, indicated in the picture, where $(x,t)$ satisfies $|x+ct| \leq 1$ or $|x-ct| \leq 1$. We ignore regions IV and V as $t<0$ there.

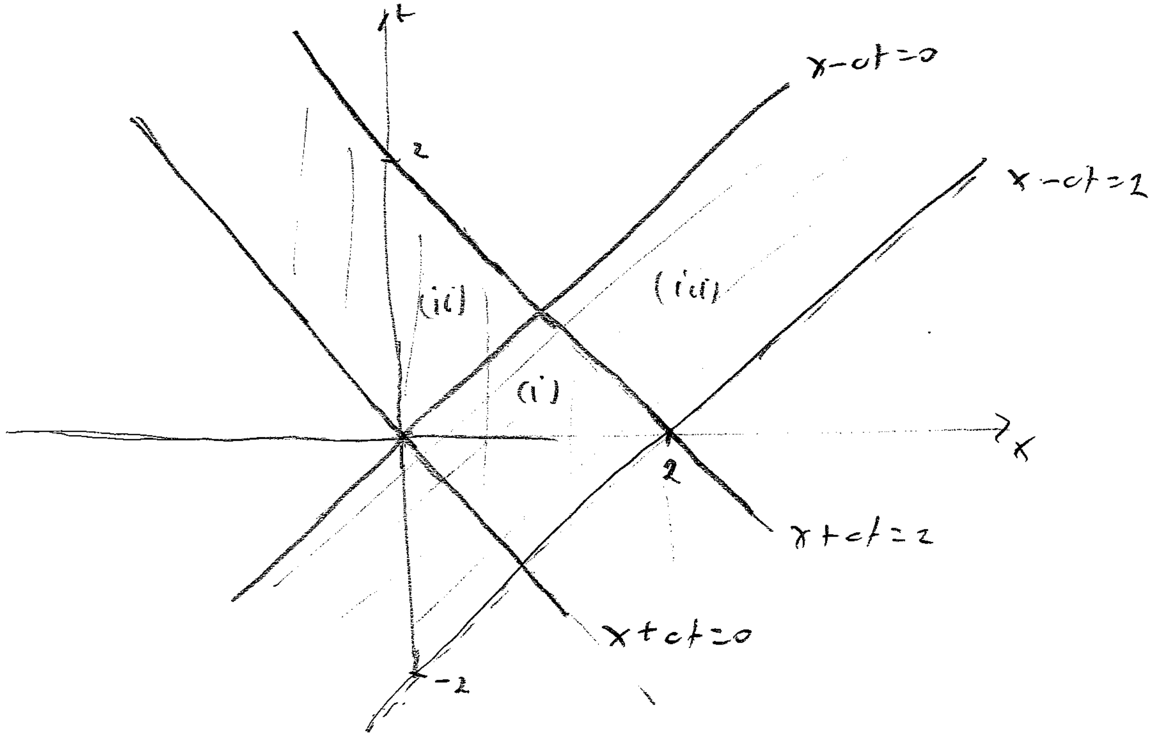
I: $$\frac{f(x+ct)+f(x-ct)}{2} = \frac{1-(x+t)^2 + 1-(x-t)^2}{2} = \frac{2-x^2-t^2-2xt-x^2-t^2+2xt}{2} = 1-x^2-t^2$$

II: $$\frac{f(x+ct)+f(x-ct)}{2} = \frac{1-(x+t)^2+0}{2} = \frac{1-x^2-t^2-2xt}{2}$$

III: $$\frac{f(x+ct)+f(x-ct)}{2} = \frac{0+1-(x-t)^2}{2} = \frac{1-x^2-t^2+2xt}{2}$$

Notice that $t=0$ gives $1-x^2$ in I, and in II and III $t>0$ so we do not test the I.C. there

For $g$, we consider $0 \le x+ct$, $x+ct \le 2$ and similarly for $x-ct$




Again, we can ignore $t \le 0$.

Notice that for the integral $\int_{x-ct}^{x+ct} g(s)\,ds$, we always have $t \ge 0$.

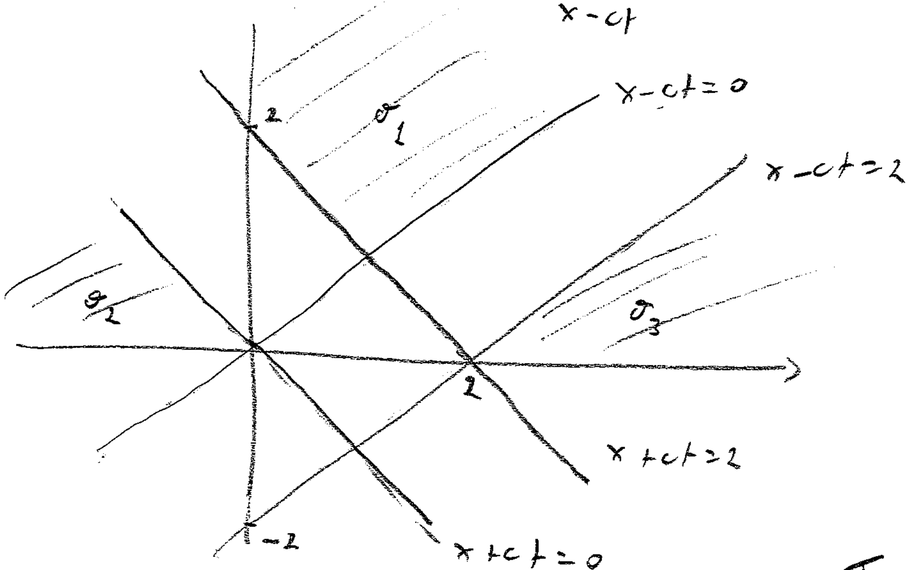
(i) $0 \le x-ct \le x+ct \le 2$, $\frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{x+t} 4\,ds = \frac{1}{2}\left(4(x+t)-4(x-t)\right) = 4t$

(ii) $x-ct \le 0 \le x+ct \le 2$ $\quad \frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_0^{x+ct} g(s)\,ds = \frac{1}{2}\int_0^{x+t} 4\,ds = 2(x+t)$

(iii) $0 \le x-ct \le 2 \le x+ct$ $\quad \frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{2} g(s)\,ds = \frac{1}{2}\int_{x-t}^{2} 4\,ds = \frac{8-4(x-t)}{2} = 4-2(x-t)$

Notice that we get $u_t(\ ) = 4$ for $t=0$ in (i).

Next, we analyze $\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$ in the outside regions:




Clearly

$$\int_{x-ct}^{x+ct} g(s)\,ds = 0$$

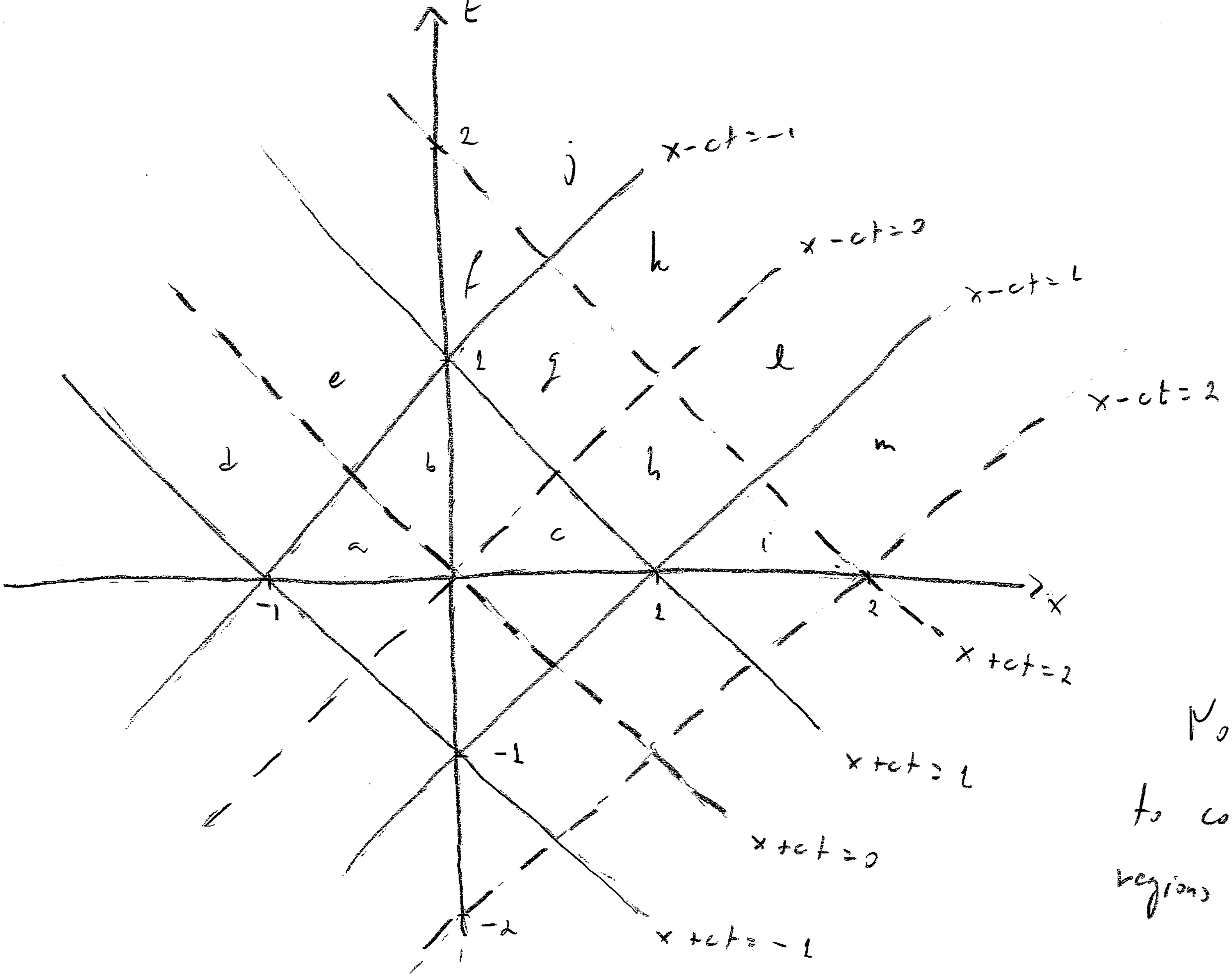
in the regions $\sigma_2$ and $\sigma_3$.

In the region $\sigma_1$, we have $x+ct \geq 2$ and $x-ct \leq 0$, thus

$(\sigma_1)$: $x-ct \leq 0 \leq 2 \leq x+ct$, so

$$\frac{1}{2c}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{0} g(s)\,ds + \frac{1}{2}\int_{0}^{2} g(s)\,ds + \int_{2}^{x+t} g(s)\,ds = 0 + \frac{1}{2}\int_{0}^{2} 4\,ds + 0$$

$$= 4.$$




Now we have to combine the regions for f and g

(a) and $\underbrace{-1 \le x+ct \text{ and } x+ct \le 1 \text{ and } -1 \le x-ct \text{ and } x-ct \le 1}_{\mathrm{I}}$ and $\underbrace{x+ct \le 0 \text{ and } x+ct \le 2 \text{ and } x-ct \le 0 \text{ and } x-ct \le 2}_{\text{not (i), (ii), or (iii)}}$ $\Rightarrow$ $-1 \le x+ct < 0$ and $-1 \le x-ct \underset{\sim}{<} 0$

have to decide on open/closed according to I.C., although here there we observe that $\delta g$ is continuous.

$$u(x,t) = 1 \underbrace{-x^2 - t^2}_{f \text{ on } \mathrm{I}} + \underbrace{0}_{?}$$

(b) $\underbrace{-1 \le x+ct \text{ and } x+ct \le 1 \text{ and } -1 \le x-ct \text{ and } x-ct \le 1}_{\mathrm{I}}$ and $0 \le x+ct$ and $x+ct \le 2$ and $\underbrace{x-ct \le 0 \text{ and } x-ct \le 2}_{\text{(ii)}}$ $\Rightarrow$ $0 \le x+ct \le 1$ and $-1 \le x-ct < 0$

$$u(x,t) = 1 - x^2 - t^2 + 2(x+t)$$

(c) $\underbrace{-1 \le x+ct \text{ and } x+ct \le 1 \text{ and } -1 \le x-ct \text{ and } x-ct \le 1}_{\mathrm{I}}$ and $0 \le x+ct$ and $x+ct \le 2$ and $\underbrace{0 \le x-ct \text{ and } x-ct \le 2}_{\text{(i)}}$ $\Rightarrow$ $0 \le x+ct \le 1$ and $0 \le x-ct \le 1$

$$u(x,t) = 1 - x^2 - t^2 + 4t$$

Note that $u(x,0) = 1 - x^2$ in (a) $\cup$ (c), $\partial_t u(x,0) = 4$ in (c), $\partial_t u = 0$ in (a) and that $u$ satisfies the equation in (a), (b), (c).

Proceeding this way we can write the solution

$$u(x,t) = \begin{cases} 1 - x^2 - t^2 & \text{for } -1 \leq x+ct < 0 \text{ and } -1 \leq x-ct < 0 \quad (a) \\ 1 - x^2 - t^2 + 2(x+t) & \text{for } 0 \leq x+ct \leq 1 \text{ and } -1 \leq x-ct < 0 \quad (b) \\ 1 - x^2 - t^2 + 4t & \text{for } 0 \leq x+ct \leq 1 \text{ and } 0 \leq x-ct \leq 1 \quad (c) \\ \vdots & \\ 0 & \text{otherwise} \end{cases}$$