Energy conservation and domain of dependence for the wave equation

We are now going to show that solutions to

$$(*)\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

conserve energy and "propagate at finite speed."

We define the energy for solutions to by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} (u_t(x,t))^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x,t)|^2\,dx \quad (= \text{kinetic + potential energy})$$

and assume that these integrals are finite. Computing:

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} u_t u_{tt} + \frac{1}{2}\int_{\mathbb{R}^n} \frac{d}{dt}(\nabla u\cdot\nabla u) = \int_{\mathbb{R}^n} u_t u_{tt} + \int_{\mathbb{R}^n} \nabla u\cdot\nabla u_t$$

where we used $\frac{d}{dt}\nabla u\cdot\nabla u = \nabla u_t\cdot\nabla u + \nabla u\cdot\nabla u_t = 2\nabla u\cdot\nabla u_t$

For any $B_R(0) \subseteq \mathbb{R}^n$: $\displaystyle\int_{B_R(0)} \nabla u \cdot \nabla u_t = -\int_{B_R(0)} \Delta u \, u_t + \int_{\partial B_R(0)} \frac{\partial u}{\partial \nu} u_t$

We assume that u and its derivatives converge to zero sufficiently fast when $|x| \to \infty$, so that $\displaystyle\lim_{R\to\infty} \int_{\partial B_R(0)} \frac{\partial u}{\partial \nu} u_t = 0$. Then:

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla u_t = \lim_{R\to\infty} \int_{B_R(0)} \nabla u \cdot \nabla u_t = -\lim_{R\to\infty} \int_{B_R(0)} \Delta u \, u_t = -\int_{\mathbb{R}^n} \Delta u_t \, u_t .$$ Thus

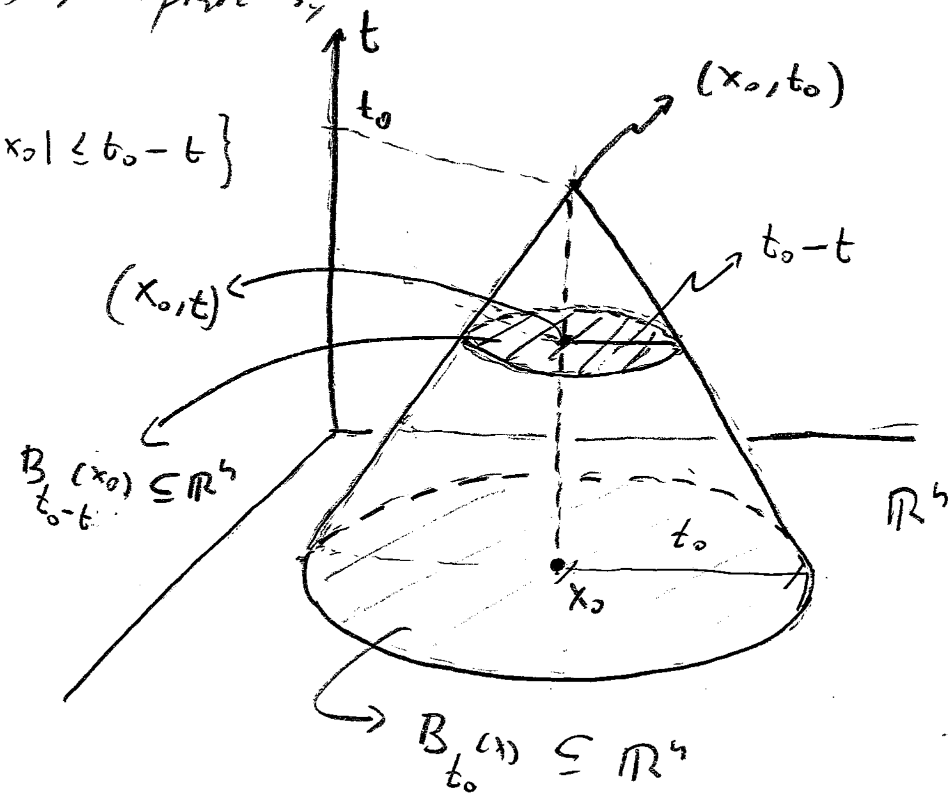
$$\frac{dE}{dt} = \int_{\mathbb{R}^n} u_t u_{tt} - \int_{\mathbb{R}^n} \Delta u \, u_t = \int_{\mathbb{R}^n} u_t \, \underbrace{(u_{tt} - \Delta u)}_{=0} = 0,$$ so $E(t)$ is constant (energy is conserved)

Thus $E(t) = E(0)$. We can compute $E(0)$:

$$E(t) = E(0) = \frac{1}{2} \int_{\mathbb{R}^n} (u_t(x,0))^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x,0)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^n} (h(x))^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla g(x)|^2 dx .$$

Now, we will show that solutions to (*) propagate at finite speed. This means the following. Given $x_0 \in \mathbb{R}^n$ and $t_0 > 0$, let $C(x_0, t_0)$ be the backwards light-cone with vertex at (x_0, t_0), defined by

$$C(x_0,t_0) = \{ (x,t) \times \mathbb{R}^n \times \mathbb{R} \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t \}$$



To say that solutions propagate at finite speed means that the value of u inside $C(x_0, t_0)$ can only be influenced by $g(x)$ and $h(x)$ for $x \in B_{t_0}(x)$ at $t = 0$.

In particular, if $g(x) = 0 = h(x)$ for all $x \in B_{t_0}(x)$ at $t = 0$, then $u(x,t) = 0$ for all $(x,t) \in C(x_0, t_0)$. (Here we took $c = 1$ in the wave equation. For $c \neq 1$, the same is true, but the cone would be defined by $|x - x_0| \le c(t_0 - t)$)

Thus, suppose that $g(x) = 0 = h(x)$ for all $x \in B_{t_0}(x_0)$. Define

$$e(t) = \frac{1}{2}\int_{B_{t_0-t}(x_0)} (u_t(x,t))^2\,dx + \frac{1}{2}\int_{B_{t_0-t}(x_0)} |\nabla u(x,t)|^2\,dx$$

(= energy inside the ball $B_{t_0-t}(x_0)$ at time t)

Then, recalling $\frac{\partial}{\partial t}\int_{\Omega(t)} f = \int_{\Omega(t)} \frac{\partial f}{\partial t} + \int_{\partial\Omega(t)} f$, we have

$$\frac{de}{dt} = \int_{B_{t_0-t}(x_0)} u_t u_{tt} - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} (u_t)^2 + \underbrace{\int_{B_{t_0-t}(x_0)} \nabla u \cdot \nabla u_t}_{\text{integrate by parts}} - \frac{1}{2}\int_{B_{t_0-t}(x_0)} |\nabla u|^2$$

$$= \int_{B_{t_0-t}(x_0)} u_t u_{tt} - \int_{B_{t_0-t}(x_0)} \Delta u\, u_t + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} (u_t^2 + |\nabla u|^2)$$

But

$$\frac{\partial u}{\partial \nu} u_t \le \left|\frac{\partial u}{\partial \nu}\right| |u_t| = \underbrace{|\nabla u \cdot \nu|}_{\le |\nabla u|\,\underbrace{|\nu|}_{=1}} |u_t| \le |\nabla u|\,|u_t| \le \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u_t^2$$

use $ab \le \frac{a^2}{2} + \frac{b^2}{2}$

Hence $$\frac{de}{dt} \le \int_{B_{t_0-t}(x_0)} u_t(\underbrace{u_{tt}-\Delta u}_{=0}) + \overbrace{\frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} (|\nabla u|^2 + u_t^2) - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} (u_t^2 + |\nabla u|^2)}^{=0}.$$ Therefore $\frac{de}{dt} \le 0$ and we conclude that $e(t) \le e(0)$, $0 \le t \le t_0$. But

$$e(0) = \frac{1}{2}\int_{B_{t_0}(x_0)} (u_t(x,0))^2\,dx + \frac{1}{2}\int_{B_{t_0}(x_0)} |\nabla u(x,t)|^2\,dx = \frac{1}{2}\int_{B_{t_0}(x_0)} h^2 + \frac{1}{2}\int_{B_{t_0}(x_0)} |\nabla g|^2 = 0$$ by our assumption on h and g.

Since on the other hand $e(t) \ge 0$, we conclude that $e(t) = 0$ for all $t \le t_0$. But this implies $u_t(x,t) = 0$ and $\nabla u(x,t) = 0$ for all $(x,t) \in C(x_0,t_0)$, since $e(t)$ is a sum of non-negative integrals of non-negative terms. This shows that $u(x,t) = 0$ inside $C(x_0,t_0)$.

Notice that g and h could be anything $B_{t_0}(x_0)$. As a consequence, we obtain the following result:

Let u and v solve $\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$ and $\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = \tilde{g} & \text{on } \mathbb{R}^n \times \{t=0\} \\ v_t = \tilde{h} & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

Suppose that $g(x) = \tilde{g}(x)$ and $h(x) = \tilde{h}(x)$ for $x \in B_{t_0}(x_0)$. Then $u = v$ inside $C(x_0, t_0)$.

In particular, <u>solutions to the wave equation are unique</u>, i.e., if $g = \tilde{g}$ and $h = \tilde{h}$, then $u = v$.

To see this, define $w = u - v$. Then w solves $\begin{cases} w_{tt} - \Delta w = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ w = w_0 & \text{on } \mathbb{R}^n \times \{t=0\} \\ w_t = w_1 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

where $w_0 = g - \tilde{g}$ and $w_1 = h - \tilde{h}$. But then $w_0(x) = 0 = w_1(x)$ for $x \in B_{t_0}(x_0)$, so by the above $w(x,t) = 0$ for $(x,t) \in C(x_0, t_0)$, so $u = v$ inside $C(x_0, t_0)$.

Finally, if $g = \tilde{g}$ and $h = \tilde{h}$, then $w_0(x) = 0 = w_1(x)$ for $x \in B_{t_0}(x_0)$ for <u>any</u> x_0 and t_0, thus $u = v$ inside any cone $C(x_0, t_0)$, hence $u = v$.

The inhomogeneous wave equation

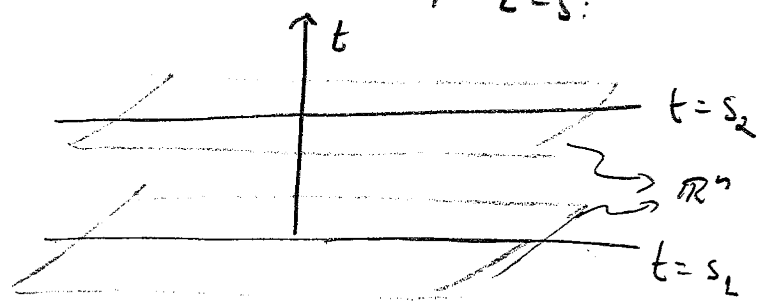
Now we will study

$$(*)\begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $f = f(x,t)$ is given.

To solve this problem, we proceed as follows. Let $v = v(x,t,s)$ be a solution to the following wave equation with initial data at $t=s$:

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ v = 0 & \text{on } \mathbb{R}^n \times \{t=s\} \\ v_t = f & \text{on } \mathbb{R}^n \times \{t=s\} \end{cases}$$



Define $u(x,t) = \int_0^t v(x,t,s)\, ds$. We claim that u solves $(*)$. Compute

$$u_t(x,t) = \partial_t\left(\int_0^t v(x,t,s)\, ds\right) = v(x,t,\underbrace{t}_{t=s}) + \int_0^t v_t(x,t,s)\, ds = \underbrace{v\big|_{t=s}}_{=0} + \int_0^t v_t(x,t,s)\, ds$$

$$u_{tt} = \partial_t \int_0^t v_t(x,t,s)\, ds = v_t(x,t,\underbrace{t}_{t=s}) + \int_0^t v_{tt}(x,t,s)\, ds = \underbrace{v_t\big|_{t=s}}_{=f(x,t)} + \int_0^t v_{tt}(x,t,s)\, ds$$

$$= f(x,t) + \int_0^t v_{tt}(x,t,s)\, ds$$

$\Delta u(x,t) = \int_0^t \Delta v(x,t,s)\,ds$. Thus $u_{tt}(x,t) - \Delta u(x,t) = f(x,t) - \int_0^t \underbrace{(v_{tt}(x,t,s) - \Delta v(x,t,s))}_{=0}\,ds$

Hence $u_{tt} - \Delta u = f$ and from the above formulas $u = 0 = \partial_t u$ at $t = 0$.

<u>Remark</u>: This method of solving a inhomogeneous PDE by solving the homogeneous PDE with different initial conditions is called the <u>Duhamel principle</u>. It can be applied to other types of PDE as well (including the heat equation).

We can now solve the full initial-value problem:
$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$
We can use the linearity of the wave equation. Let v and w solve

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = g & \text{on } \mathbb{R}^n \times \{t=0\} \\ v_t = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}, \quad \begin{cases} w_{tt} - \Delta w = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ w = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \\ w_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

each being a problem that we learned how to solve. Then $u = v + w$ solves the original initial-value problem.