We now move to establish (iii). Since we showed u to be C^2, we can apply Δ to it. We find

$$\Delta u(x) = \int_{\mathbb{R}^n} \Gamma(y)\, \Delta_x f(x-y)\, dy$$ where we use the notation Δ_x to indicate that the derivatives in Δ are with respect to x.

Let $\varepsilon > 0$ and consider $B_\varepsilon(0)$. We can split the integral as

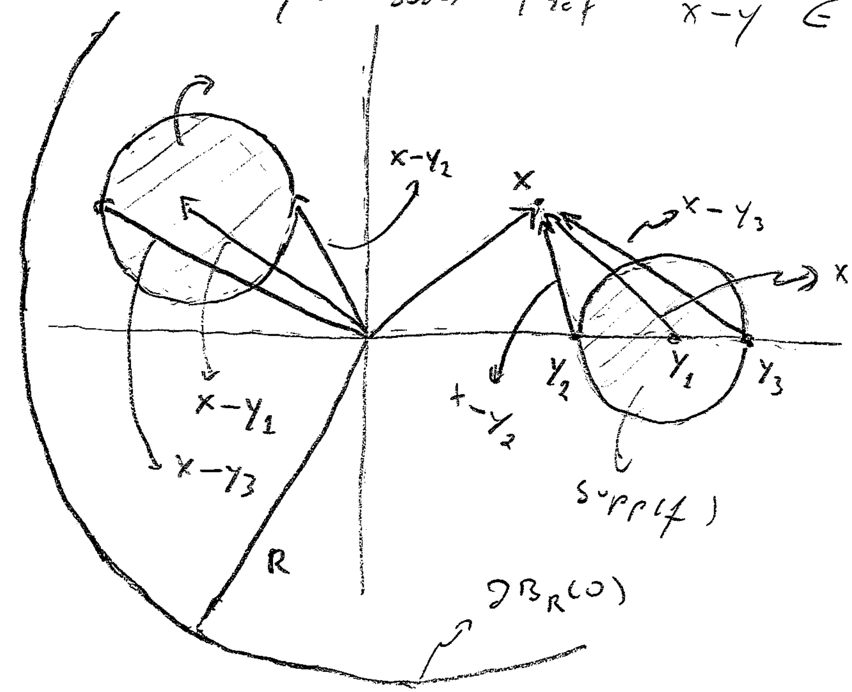
$$\Delta u(x) = \int_{\mathbb{R}^n} \Gamma(y)\, \Delta_x f(x-y)\, dy = \lim_{\varepsilon \to 0} \left[\int_{B_\varepsilon(0)} \Gamma(y)\, \Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y)\, \Delta_x f(x-y)\, dy \right]$$

Since $\Delta_x f(x-y)$ is continuous and has compact support (because $f \in C_c^2(\mathbb{R}^n)$), we know that $|\Delta_x f(x-y)| \leq M$ for some constant M and all $x, y \in \mathbb{R}^n$. Thus

$$\left| \int_{B_\varepsilon(0)} \Gamma(y)\, \Delta_x f(x-y)\, dy \right| \leq M \int_{B_\varepsilon(0)} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|y|^{n-2}}\, dy = \frac{M}{n(n-2)\alpha(n)} \int_0^\varepsilon \int_{\partial B_1(0)} d\Omega\, \frac{r^{n-1}}{r^{n-2}}\, dr$$

$= \text{constant} \cdot \varepsilon^2$. Thus $\lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \Gamma(y)\, \Delta_x f(x-y)\, dy = 0$.

For the second integral, note that $f(x-y) \neq 0$ only when $x-y \in \operatorname{supp}(f)$. Thus, when we integrate over y, $\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} I(y)\, \Delta_x f(x-y)\, dy$ will not be necessarily zero only for those y's such that $x-y \in \operatorname{supp}(f)$, or equivalently for $y \in x - \operatorname{supp}(f)$



Therefore, we can choose R so large that

$$x - \operatorname{supp}(f) \subseteq B_R(0)$$

and $f(x-y)$ and its derivatives vanish on $\partial B_R(0)$. This is possible because f has compact support.

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} I(y)\, \Delta_x f(x-y)\, dy = \int_{B_R(0) \setminus B_\varepsilon(0)} I(y)\, \Delta_x f(x-y)\, dy$$

Remark: The picture is merely illustrative, $\operatorname{supp}(f)$ need not to be a ball, x and zero may belong to $\operatorname{supp}(f)$ or $x - \operatorname{supp}(f)$

Next, notice that $\Delta_x f(x-y) = \Delta_y f(x-y)$.

To see this, denote

$$z = x - y$$

Compute $\dfrac{\partial}{\partial x_i}(f(z)) = \displaystyle\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\frac{\partial z_j}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) \overbrace{\frac{\partial}{\partial x_j}(x_i - y_i)}^{=\begin{cases} 0, & j \neq i \\ 1, & j = i\end{cases}} = \frac{\partial f}{\partial z_i}(z)$

Taking another derivative we find $\dfrac{\partial^2}{\partial x_i^2}(f(z)) = \dfrac{\partial^2 f}{\partial z_i^2}(z)$. Thus $\Delta_x(f(z)) = \Delta_z f(z)$.

On the other hand, $\dfrac{\partial}{\partial y_i}(f(z)) = \displaystyle\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\frac{\partial z_j}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) \underbrace{\frac{\partial}{\partial y_i}(x_j - y_j)}_{=\begin{cases} 0, & j \neq i \\ -1, & j = 1\end{cases}} = -\frac{\partial f}{\partial z_i}(z)$

Taking another derivative, we get another negative, thus

$$\Delta_y(f(z)) = \Delta_z f(z) = \Delta_x(f(z)),$$

as claimed. Therefore,

we write $dS(y)$ to indicate that the variable being integrated on the boundary is y

$$\int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy = \int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_y f(x-y)\,dy \underset{\substack{\text{integrate}\\ \text{by parts}}}{=} -\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\,dy + \int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)$$

where we use ∇_y to denote the gradient on the y-variable. Since $\partial(B_R(0)\setminus B_\varepsilon(0))$ $= \partial B_R(0) \cup \partial B_\varepsilon(0)$ (see picture in the previous page) we have

$$\int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y) = \int_{\partial B_R(0)} \Gamma(y)\,\underbrace{\nabla_y f(x-y)}_{=0 \text{ on } \partial B_R(0)}\cdot\nu\,dS(y)\ \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y) = \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)$$

Therefore

$$\int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_y f(x-y)\,dy = -\overbrace{\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\,dy}^{\text{integrate by parts again}} + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)$$

$$= -\left[-\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\cdot\nabla\Gamma(y)\,f(x-y)\,dy + \int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \nabla\Gamma(y)\cdot\nu\,f(x-y)\,dS(y)\right] + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y).$$

But $\nabla\cdot\nabla\Gamma(y) = \Delta\Gamma(y) = 0$ for $y \neq 0$ (recall that Γ solves Laplace's equation except at the origin), thus the first integral vanishes. The second integral becomes an integral only over $\partial B_\varepsilon(0)$ by the same argument used above. Therefore:

$$\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy = -\int_{\partial B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nu\,f(x-y)\,dS(y) + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y).$$

We now want to take the limit $\varepsilon \to 0$. Look first at the second integral:

$$\left|\int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)\right| \leq \int_{\partial B_\varepsilon(0)} |\Gamma(y)|\,|\nabla_y f(x-y)\cdot\nu|\,dS(y).$$

Recall the following property of the dot product: $|v\cdot u| \leq |v||u|$.
(called Cauchy-Schwarz inequality)

Then $|\nabla_y f(x-y)\cdot \nu| \le |\nabla_y f(x-y)|\underbrace{|\nu|}_{=1} = |\nabla_y f(x-y)|$. Since $f \in C_c^2(\mathbb{R}^n)$, ∇f is continuous and has compact support, thus by fact 1 there exists a constant M such that $|\nabla f(z)| \le M$ for all $z \in \mathbb{R}^n$. Hence

$$\int_{\partial B_\varepsilon(0)} |\Gamma(y)|\,|\nabla_y f(x-y)\cdot\nu|\,dS(y) \le \frac{M}{n(n-2)\alpha(n)} \int_{\partial B_\varepsilon(0)} \frac{1}{|y|^{n-2}}\,dS(y) = \frac{M}{n(n-2)\alpha(n)} \int_{\partial B_\varepsilon(0)} \frac{1}{|y|^{n-2}}\,dS(y)$$

On $\partial B_\varepsilon(0)$, $|y| = \varepsilon$ and $dS(y) = \varepsilon^{n-1}\,d\Omega$, so we get

$$\frac{M}{n(n-2)\alpha(n)} \int_{\partial B_1(0)} \frac{1}{\varepsilon^{n-2}}\,\varepsilon^{n-1}\,d\Omega = \text{constant}\cdot\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon \to 0, \text{ so}$$

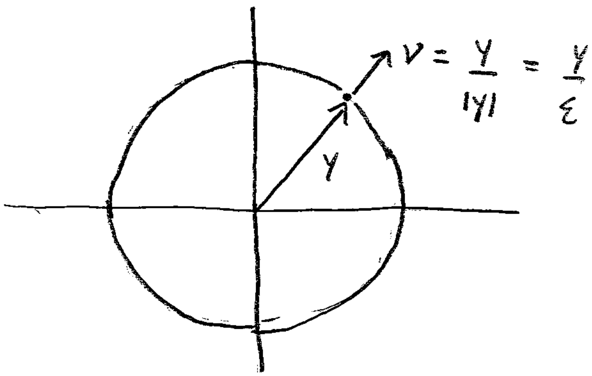
$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y) = 0.$$ It remains to analyze

$$-\int_{\partial B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nu\, f(x-y)\,dS(y).$$

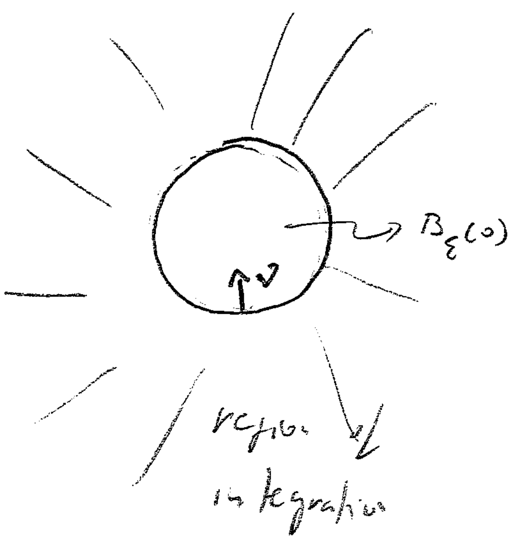
Recall that we computed $\dfrac{\partial \Gamma(y)}{\partial y_i} = -\dfrac{1}{n\alpha(n)} \dfrac{y_i}{|y|^n}$, therefore we have

$$\nabla \Gamma(y) = \left(\frac{\partial \Gamma(y)}{\partial y_1}, \dots, \frac{\partial \Gamma(y)}{\partial y_n} \right) = \left(-\frac{1}{n\alpha(n)} \frac{y_1}{|y|^n}, \dots, -\frac{1}{n\alpha(n)} \frac{y_n}{|y|^n} \right) = -\frac{1}{n\alpha(n)} \frac{1}{|y|^{n-1}} \left(\frac{y_1}{|y|}, \dots, \frac{y_n}{|y|} \right)$$

Since on $\partial B_\varepsilon(0)$ $|y| = \varepsilon$, we can write $\nabla \Gamma(y) = -\dfrac{1}{n\alpha(n)} \dfrac{1}{\varepsilon^{n-1}} \left(\dfrac{y_1}{\varepsilon}, \dots, \dfrac{y_n}{\varepsilon} \right)$.



Notice that $\left(\dfrac{y_1}{\varepsilon}, \dots, \dfrac{y_n}{\varepsilon} \right) = \dfrac{y}{\varepsilon}$ is the unit outer normal to $\partial B_\varepsilon(0)$. However, in our case the unit outer normal points to the inside of the ball because the original domain of integration was exterior to $B_\varepsilon(0)$, thus $\dfrac{y}{\varepsilon} = -\nu$. Hence



$$\nabla \Gamma(y) = \frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}} \nu. \quad \text{Therefore}$$

$$\nabla \Gamma(y) \cdot \nu = \frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}} \nu \cdot \nu = \frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}}$$

We have

$$-\int_{\partial B_\varepsilon(0)} \nabla \Gamma(y)\cdot \nu\, f(x-y)\, dS(y) = -\frac{1}{n\alpha(n)}\frac{1}{\varepsilon^{n-1}}\int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y).$$ Changing variables $x-y=z$

and recalling that $\mathrm{vol}(\partial B_\varepsilon(0)) = \mathrm{vol}(\partial B_\varepsilon(x)) = n\alpha(n)\varepsilon^{n-1}$ we obtain

$$-\frac{1}{\mathrm{vol}(\partial B_\varepsilon(x))}\int_{\partial B_\varepsilon(x)} f(z)\, dS(z) \longrightarrow -f(x) \text{ as } \varepsilon\to 0$$ by fact 3.

Putting all together we have

$$\Delta u(x) = \lim_{\varepsilon\to 0}\left[\int_{B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\, dy\right] = -f(x),$$

finishing the proof. $\blacksquare$

<u>Remark</u> Since $\Delta \Gamma(y) = 0$ for all $y \neq 0$, we might be tempted to do the following.

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy \implies \Delta u(x) = \int_{\mathbb{R}^n} \Delta_x \Gamma(x-y) f(y)\, dy .$$

Now, we could say that since the value of an integral remains unchanged if we remove a point from the domain of integration, then what happens at $y=0$ doesn't matter, and since $\Delta \Gamma(y) = 0$ for all $y \neq 0$, we should have $\Delta u(x) = \int_{\mathbb{R}^n} \underbrace{\Delta_x \Gamma(x-y)}_{=0 \;\; x \neq y} f(y)\, dy = 0$, and not $\Delta u(x) = -f(x)$ as we just proved.

This argument, however, is wrong for several reasons:

1. We can only switch the order $\Delta_x \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Delta_x \Gamma(x-y) f(y)\, dy$ the integral of $\Delta_x \Gamma(x-y) f(y)$ is finite. This, however, will not not to be the case (except in some trivial cases such as $f=0$). To see this, recall that we computed $\partial_i \partial_j \Gamma$, from which we can compute $\Delta_x \Gamma(x-y)$. The resulting expression says essentially that $\Delta_x \Gamma(x-y) \sim \dfrac{1}{|x-y|^n}$.

Integrating in polar coordinates then will give $\int_{\mathbb{R}^n} \frac{1}{|x-y|^n} f(y)\, dy = \int_{\mathbb{R}^n} \frac{1}{|z|^n} f(x-z)\, dz$

$$= \underbrace{\int_{B_R(0)}}_{\text{because } f \in C_c^2(\mathbb{R}^n)} \frac{1}{r^n} f(x-y)\, r^{n-1}\, dr\, d\Omega \geq \underbrace{(\text{minimum of } f)}_{\substack{\text{which exists} \\ \text{because } f \in C_c^2(\mathbb{R}^n)}} \underbrace{\int_{\partial B_1(0)} d\Omega}_{= n\,\omega_n} \underbrace{\int_0^R \frac{1}{r}\, dr}_{\ln r \big|_0^R} = +\infty$$

(except for trivial cases of f).

That's why we have to remove from the domain of integration a ball of radius ε about x (which, after a change of coordinates, became $B_\varepsilon(0)$) and then take the limit $\varepsilon \to 0$.

<u>Notation</u>: We see that the quantity $\nabla f(x) \cdot \nu(x)$, $x \in \partial\Omega$ (Ω a domain) appears often in the calculations. Since $\nabla f \cdot \nu$ is the rate of change of f in the direction normal to $\partial\Omega$, we call it <u>normal derivative</u> and denote by $\frac{\partial f}{\partial \nu}$.