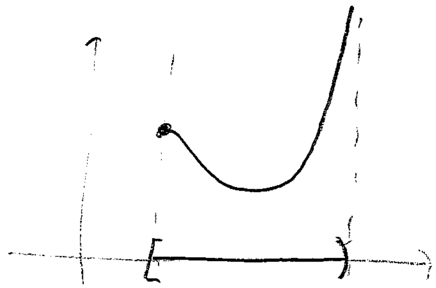
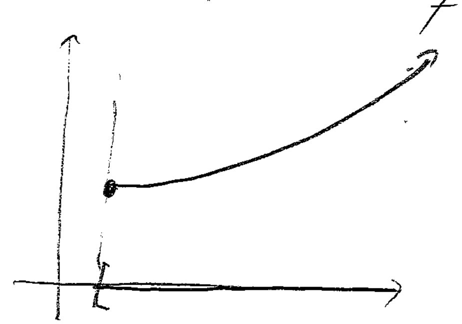
**Remark**: we will prove the theorem for $n \geq 3$, leaving the case $n = 2$ as exercise.

In order to prove the theorem, we will need the following facts.

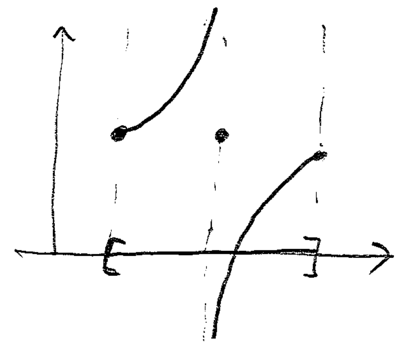
**Fact 1**: Let $K$ be a closed and bounded set in $\mathbb{R}^n$. Bounded means that $K$ is contained in a ball of radius $R$ and center zero for some $R$, i.e., $K \subseteq B_R(0)$. Let $f : K \to \mathbb{R}$ be continuous. Then $f$ attains a maximum and a minimum within $K$, thus there exists constants a constant $M$ such that $|f(x)| \leq M$ for all $x \in K$. The following pictures illustrate this fact:



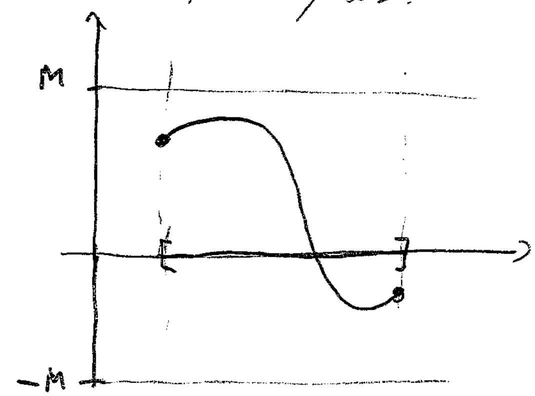
$K$ bounded but not closed
$f$ continuous
Fact 1 not true



$K$ closed but not bounded
$f$ continuous
Fact 1 not true



$K$ closed and bounded
$f$ not continuous
Fact 1 not true




$K$ closed and bounded
$f$ continuous.

Similar pictures can be drawn in higher dimensions

<u>Fact 2</u> Let $B_R(0)$ be the ball of radius $R$ centered at the origin in $\mathbb{R}^n$.
The volume element $dx$ ($= dV$ in 3d calculus) in $B_r(0)$ can be written as

$dx = r^{n-1}\, dr\, d\Omega$, where $d\Omega$ is the volume element on $\partial B_1(0)$.

<u>2d</u>: $dx = r\, dr \underbrace{d\theta}_{d\Omega}$ ; <u>3d</u> $dx = r^2 \sin\varphi\, dr\, d\varphi\, d\theta = r^2 dr \underbrace{\sin\varphi\, d\varphi\, d\theta}_{d\Omega}$

It follows that the volume element $dS$ on $\partial B_r(0)$ is $dS = r^{n-1}\, d\Omega$

<u>2d</u> $dS = r\, d\theta$ <u>3d</u> $dS = r^2 \sin\varphi\, d\varphi\, d\theta$

<u>Fact 3</u> Let $f$ be a $C^2$ function then

$$\underbrace{\frac{1}{\mathrm{vol}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} f\, dS}_{\text{average of } f \text{ over } \partial B_r(x_0)} \longrightarrow f(x_0) \quad \text{as } r \to 0.$$

Note that $\mathrm{vol}(\partial B_r(x_0)) = n\,\alpha(n)\, r^{n-1}$

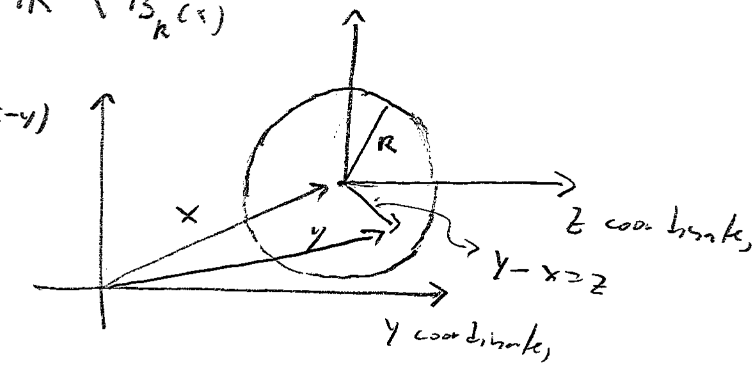
proof of the theorem:

To show that $u$ is well-defined, fix $x \in \mathbb{R}^n$ and $R > 0$ (say, $R = 1$), and write

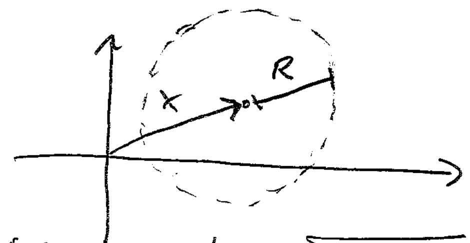
$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_{B_R(x)} \Gamma(x-y) f(y)\, dy + \int_{\mathbb{R}^n \setminus B_R(x)} \Gamma(x-y) f(y)\, dy$$

Change variables in the first integral: $z = y - x = -(x-y)$

$$\int_{B_R(x)} \Gamma(x-y) f(y)\, dy = \int_{B_R(0)} \Gamma(-z) f(z+x)\, dz ,$$




As $z$ varies within $B_R(0)$, $z + x$ varies within $B_{R+|x|}(0)$.




Since $B_{R+|x|}(0) \subset \overline{B_{2(R+|x|)}(0)}$ and $f$ attains a maximum and a minimum within $\overline{B_{2(R+|x|)}(0)}$, we have that $|f(z+x)| \leq M$ for some $M$ and all $z \in B_R(0)$.

Next, $$\Gamma(-z) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|-z|^{n-2}} = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|z|^{n-2}}$$

Hence: $\quad |\Gamma(-z)| = \Gamma(-z)$ since $\Gamma \geq 0$

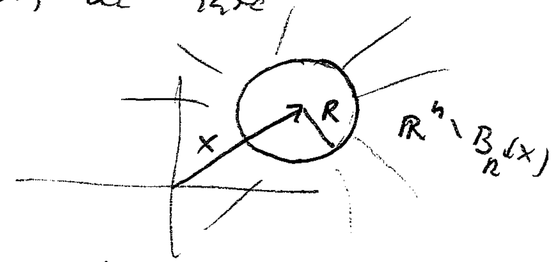
$$\left| \int_{B_R(0)} \Gamma(-z) f(z+x)\,dz \right| \leq \int_{B_R(0)} \widetilde{\Gamma(z)}\, |f(z+x)|\,dz \leq M \int_{B_R(0)} \Gamma(z)\,dz = \frac{M}{n(n-2)\alpha(n)} \int_{B_R(0)} \frac{1}{|z|^{n-2}}\,dz$$

Integrating in polar coordinates: $\displaystyle\int_{B_R(0)} \frac{1}{|z|^{n-2}}\,dz = \int_0^R \int_{\partial B_r(0)} \frac{1}{r^{n-2}}\,dS = \int_0^R \frac{1}{r^{n-2}} \int_{\partial B_1(0)} r^{n-1}\,d\Omega$

$\displaystyle = n\alpha(n) \int_0^R r\,dr = \frac{n\alpha(n)}{2} R^2$. Thus $\displaystyle\left| \int_{B_R(x)} \Gamma(x-y) f(y)\,dy \right| < \infty$.

For the second integral, note that for any $y \in \mathbb{R}^n \setminus B_R(0)$ we have

$$\Gamma(x-y) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x-y|^{n-2}} \leq \frac{1}{n(n-2)R^{n-2}} \quad \text{since } |x-y| \geq R$$




Since $f$ has compact support and is continuous, we have $|f(y)| \leq M$ for all $y \in \mathbb{R}^n$, thus $\displaystyle\left| \int_{\mathbb{R}^n \setminus B_R(x)} \Gamma(x-y) f(y)\,dy \right| \leq \int_{\mathbb{R}^n \setminus B_R(x)} \Gamma(x-y)\,|f(y)|\,dy < \infty$, hence $|u(x)| < \infty$, showing (i).

<u>Remark</u>: we could have used $|f(y)| \leq M$, $y \in \mathbb{R}^n$, in the estimate of the first

integral, thus avoiding the argument with $B_{|x|+R}(0)$ etc. But we think it is useful to show that the first integral is $< \infty$ without using that $f$ has compact support, as that trick is generally useful.

To show (ii), change variables:

$$u(x) = \int_{\mathbb{R}^n} \Gamma(\underbrace{x-y}_{z}) f(y)\, dy = \int_{\mathbb{R}^n} \Gamma(z) f(x-z) \overbrace{\left| \det \left( \frac{\partial z_i}{\partial y_j} \right) \right|}^{=1} dz = \int_{\mathbb{R}^n} \Gamma(y) f(x-y)\, dy$$

For $h \neq 0$ and $e_i = (0,0,\ldots,\underset{\substack{\downarrow \\ i^{th} \text{ component}}}{1},\ldots,0,0)$, we have

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Gamma(y) \frac{f(x+he_i-y)}{h} dy - \int_{\mathbb{R}^n} \Gamma(y) \frac{f(x-y)}{h} dy = \int_{\mathbb{R}^n} \Gamma(y) \left( \frac{f(x+he_i-y) - f(x-y)}{h} \right) dy$$

But $\dfrac{f(x+he_i-y) - f(x-y)}{h} \to \dfrac{\partial f}{\partial x_i}(x-y)$ uniformly on $\mathbb{R}^n$ as $h \to 0$. Since $\dfrac{\partial f}{\partial x_i}$ is continuous, the same argument as in (i) shows that $\int_{\mathbb{R}^n} \Gamma(y) \frac{\partial f}{\partial x_i}(x-y)\, dy < \infty$, thus $\dfrac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \dfrac{u(x+he_i) - u(x)}{h}$ exists and

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Gamma(y) \frac{\partial f}{\partial x_i}(x-y)\, dy.$$

Using that $\frac{\partial f}{\partial x_i}$ is continuous (because $f$ is $C^2$), we can then show that $\frac{\partial u}{\partial x_i}$ is continuous, thus $u \in C^1(\mathbb{R}^n)$.

To show that $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exists, we repeat the above arguments applied to $\frac{\partial u}{\partial x_i}$.

To show that $\frac{\partial^2 u}{\partial x_i \partial x_j}$, we then use the continuity of $\frac{\partial^2 f}{\partial x_i \partial x_j}$. This establishes (i).

Exercise: complete the proof of (i) by developing the above arguments.