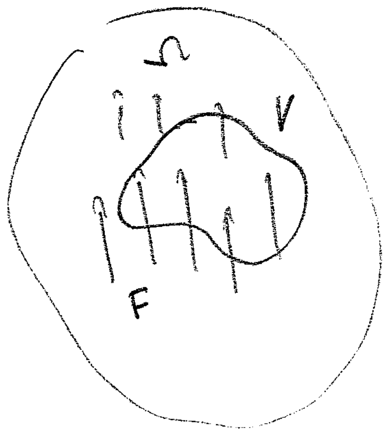
Laplace's equation and Poisson's equation

We will investigate Laplace's equation $\Delta u = 0$ and Poisson's equation $-\Delta u = f$, both in a domain $\Omega \subseteq \mathbb{R}^n$. In both cases the unknown is $u = u(x_1, \dots, x_n)$. The negative sign on $-\Delta u$ is purely conventional, and solving $-\Delta u = f$ is of course equivalent to solving $\Delta u = \tilde{f}$, $\tilde{f} = -f$.

Laplace's equation appears in many applications. For example, suppose we have the density of some quantity u (e.g. a chemical concentration) in equilibrium in Ω. If V is any subregion of Ω, the net flux of u through ∂V must be zero. Denoting the flux by $F = (F_1, F_2, \dots, F_n)$ we have



$$\int_{\partial V} F \cdot \nu \, ds = 0.$$

Typically, the flux is proportional to the gradient of u but points in the opposite direction (the flow is from regions of higher to lower concentration): $F = -k \nabla u$, Using the divergence theorem:

$$0 = \int_{\partial V} F \cdot \nu \, dS = \int_V \operatorname{div} F \, dx = \int_V -k \operatorname{div} \nabla u \, dx, \quad \text{but}$$

$$\operatorname{div} \nabla u = \operatorname{div}\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = \Delta u, \quad \text{thus}$$

$\int_V \Delta u \, dx = 0$. Since this must hold for any region V, we find that:

$$\Delta u = 0 \quad \text{in } \Omega$$

Solutions to Laplace's equation can also be tought as stationary solutions to the heat or wave equation.

Poisson's equation also appears in many applications. For instance, from Maxwell's equations $\operatorname{curl} E = 0$ and $\operatorname{div} E = \rho$, ρ = charge density, we have (from $\operatorname{curl} E = 0$) that $E = -\nabla \phi$, where ϕ is known as the electric potential. Thus $\operatorname{div} E = -\operatorname{div} \nabla \phi = \rho$ i.e.

$$-\Delta \phi = \rho$$

One important property of Laplace's equation is that it is <u>invariant under rotations</u>, meaning the following. A $n \times n$ matrix M is called an <u>orthogonal matrix</u> if $MM^T = I$, where M^T is the transpose of M and I is the identity matrix (thus, orthogonal matrices are always invertible and $M^{-1} = M^T$). If $x \in \mathbb{R}^n$, we think of of Mx as the vector obtained from x by rotating it about some axis (which depends on M). For instance, a rotation of θ about the z-axis in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose that u solves $\Delta u = 0$. Define a new function $\tilde{u}$ by $\tilde{u}(x) = u(Mx)$. Then $\Delta \tilde{u} = 0$.

Since rotations do not destroy the property of being a solution to Laplace's equation, we attempt to solve it by seeking rotationally symmetric solutions.

Take $\Omega = \mathbb{R}^n$, set $u(x) = \sigma(r)$ where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
We will try to select σ so that it solves Laplace's equation. Note that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \frac{2x_i}{\sqrt{x_1^2 + \cdots + x_n^2}} = \frac{x_i}{r} \quad \text{provided that } r \neq 0.$$

Then $\dfrac{\partial u}{\partial x_i} = \dfrac{\partial \sigma}{\partial r}\dfrac{\partial r}{\partial x_i} = \sigma' \dfrac{x_i}{r} \quad \left(\sigma' = \dfrac{\partial \sigma}{\partial r}\right)$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\sigma' \frac{x_i}{r}\right) = \frac{\partial \sigma'}{\partial x_i}\frac{x_i}{r} + \sigma' \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{\partial \sigma'}{\partial r}\frac{\partial r}{\partial x_i}\frac{x_i}{r} + \sigma' \frac{r\frac{\partial x_i}{\partial x_i} - x_i \frac{\partial r}{\partial x_i}}{r^2}$$

$$= \sigma'' \frac{x_i^2}{r^2} + \frac{\sigma'}{r^2}\left(r - \frac{x_i^2}{r}\right) = \sigma'' \frac{x_i^2}{r^2} + \sigma'\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$ Therefore:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n}\left(\sigma'' \frac{x_i^2}{r^2} + \sigma'\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right)$$

$$= \frac{\sigma''}{r^2}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}\frac{\sigma'}{r} - \frac{\sigma'}{r^3}\sum_{i=1}^{n} x_i^2 = \frac{\sigma''}{r^2} r^2 + n\frac{\sigma'}{r} - \frac{\sigma'}{r^3}\cdot r^2 = \sigma'' + \frac{n-1}{r}\sigma',$$

i.e.,

$$\Delta u = \sigma'' + \frac{n-1}{r}\sigma'$$

Since we want $\Delta u = 0$, we need to solve the following ODE:

$v'' + \frac{n-1}{r} v' = 0$. If $v' \neq 0$, this can be written as $\frac{v''}{v'} = \frac{1-n}{r}$.

But $\frac{v''}{v'} = (\ln(|v'|))'$, so $(\ln(|v'|))' = \frac{1-n}{r}$. Integrating both sides gives $\ln|v'| = \int \frac{1-n}{r} dr = (1-n)\ln r + A = A + \ln r^{1-n}$, so $|v'| = e^A e^{\ln r^{1-n}}$

We can remove the absolute value and absorb the $\pm$ sign in the constant e^A, which we relabel a, so $v' = \frac{a}{r^{n-1}}$. We integrate again to find v:

$$v(r) = \begin{cases} b \ln r + c & \text{if } n = 2 \\ \frac{b}{r^{n-2}} + c & \text{if } n \geq 3 \end{cases}$$ where b and c are constants.

Note that $u(x) = v(r)$ is not quite a solution to Laplace's equation in $\mathbb{R}^n$, since it is not defined at $x = 0$ ($r = 0$). However, we will see how it can be used to construct actual solutions.

We will make a specific choice for the constants b and c which will be convenient in the ensuing calculations.

Def. The function $\Gamma : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ given by

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| , & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} , & n \geq 3 \end{cases}$$

is called the <u>Fundamental Solution to Laplace's equation</u>

Here, $\alpha(n)$ is the volume of the ball of radius one and center zero in $\mathbb{R}^n$ (so $\alpha(3) = \frac{4}{3}\pi$).

We are going to need the quantities $\partial_i \Gamma$ and $\partial_i \partial_j \Gamma$. Let's compute, for $n \geq 3$:

$$\partial_i \Gamma(x) = \frac{1}{n(n-2)\alpha(n)} \partial_i |x|^{2-n} = \frac{2-n}{n(n-2)\alpha(n)} |x|^{1-n} \overbrace{\partial_i |x|}^{= \partial_i r = \frac{x_i}{r} = \frac{x_i}{|x|}} = -\frac{1}{n\alpha(n)} \frac{x_i}{|x|^n}$$

$$\partial_j \partial_i \Gamma(x) = -\frac{1}{n\alpha(n)} \partial_j \frac{x_i}{|x|^n} = -\frac{1}{n\alpha(n)} \frac{|x|^n \partial_j x_i - x_i \partial_j |x|^n}{|x|^{2n}}.$$ We have that

$\partial_j x_i = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}$, thus we introduce the symbol δ_{ij} defined by $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$,

so that $\partial_j x_i = \delta_{ij}$. $\partial_j |x|^n = n|x|^{n-1} \partial_j |x| = n|x|^{n-1} \frac{x_j}{|x|} = n|x|^{n-2} x_j$. Hence:

$$\partial_j \partial_i \Gamma(x) = -\frac{1}{n\alpha(n)} \frac{1}{|x|^{2n}} \left(|x|^n \delta_{ij} - n|x|^{n-2} x_i x_j \right)$$

$$= -\frac{1}{n\alpha(n)} \left(|x|^2 \delta_{ij} - n x_i x_j \right) \frac{1}{|x|^{n+2}}$$

Summing up

$$\partial_i \Gamma(x) = -\frac{1}{n\alpha(n)} \frac{x_i}{|x|^n} \quad \text{and} \quad \partial_i \partial_j \Gamma(x) = -\frac{1}{n\alpha(n)} \left(|x|^2 \delta_{ij} - n x_i x_j \right) \frac{1}{|x|^{n+2}}$$

We are going to also need estimates for $\partial_i \Gamma$ and $\partial_{ij}^2 \Gamma$. For this, observe that $\left| \frac{x_i}{|x|^n} \right| = \frac{|x_i|}{|x|^n} = \frac{\sqrt{x_i^2}}{|x|^n} \leq \frac{\sqrt{x_1^2 + \dots + x_n^2}}{|x|^n} = \frac{1}{|x|^{n-1}}$ and

$$\left| \frac{x_i x_j}{|x|^{n+2}} \right| = \frac{|x_i| |x_j|}{|x|^{n+2}} = \frac{\sqrt{x_i^2} \sqrt{x_j^2}}{|x|^{n+2}} \leq \frac{\sqrt{x_1^2 + \dots + x_n^2} \sqrt{x_1^2 + \dots + x_n^2}}{|x|^{n+2}} = \frac{1}{|x|^n}$$

Thus

$$|\partial_i \Gamma(x)| \leq \frac{1}{n\alpha(n)} \frac{1}{|x|^{n-1}}, \text{ and}$$

$$|\partial_i \partial_j \Gamma(x)| = \frac{1}{n\alpha(n)} \left| \frac{|x|^2}{|x|^{n+2}} \delta_{ij} - n \frac{x_i x_j}{|x|^{n+2}} \right| \leq \frac{1}{n\alpha(n)} \left(\frac{\delta_{ij}}{|x|^n} + n \frac{|x_i x_j|}{|x|^{n+2}} \right)$$

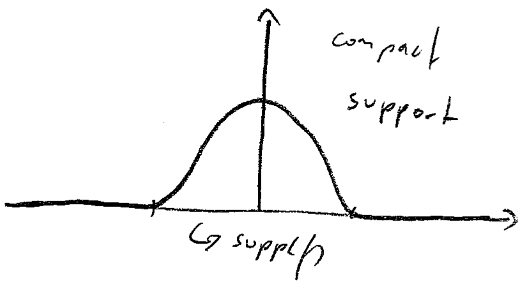
$$\leq \frac{1}{n\alpha(n)} \left(\frac{1}{|x|^n} + \frac{n}{|x|^n} \right) = \frac{n+1}{n\alpha(n)} \frac{1}{|x|^n}$$

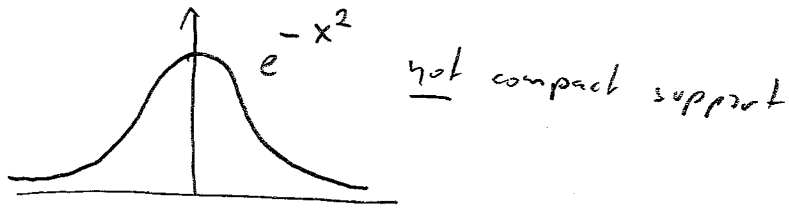
Summing up: $|\partial_i \Gamma(x)| \leq \frac{1}{n\alpha(n)} \frac{1}{|x|^{n-1}}$, $|\partial_i \partial_j \Gamma(x)| \leq \frac{n+1}{n\alpha(n)} \frac{1}{|x|^n}$

Analogous estimates hold in 2d.

Def. Let $f:\mathbb{R}^n \to \mathbb{R}$ and set $A = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$. We define the support of f, denoted $\mathrm{supp}(f)$, as the set $\mathrm{supp}(f) = \bar{A}$. If $\bar{A} \subseteq B_R(0)$ (= ball of radius R and center zero) for some R, we say that f has compact support. The space of C^k functions with compact support is denoted by $C_c^k(\mathbb{R}^n)$

Remark. If $f \in C_c^k(\mathbb{R}^n)$, then $f(x) = 0$ for all x such that $|x|$ is sufficiently large.





Theorem (solution to Poisson's equation) Let $f:\mathbb{R}^n \to \mathbb{R}$ be a C^2 function with compact support. Set

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y) \quad (\text{i.e., } u = \Gamma * f)$$

Then

(i) u is well-defined

(ii) $u \in C^2(\mathbb{R}^n)$

(iii) $-\Delta u = f$.