Laplace transform

Let us denote $\mathbb{R}_+ = (0, \infty)$.

Def. If $u : \mathbb{R}_+ \to \mathbb{R}$ is such that $\int_0^\infty |u(t)| dt < \infty$, we define its Laplace transform, denoted $u^\#$ (u-sharp) or $\mathcal{L}\{u\}$, is the function $u^\# : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$u^\#(s) = \int_0^\infty e^{-st} u(t)\, dt \qquad (s \geq 0)$$

Remark: since $\left| \int_0^\infty e^{-t} u(t)\, dt \right| \leq \int_0^\infty |e^{-st}|\, |u(t)|\, dt \underset{s \geq 0}{\leq} \int_0^\infty |u(t)| dt < \infty$, $u^\#$ is well defined.

But as we did with the Fourier transform, we will suppose that all our integrals converge and can be manipulated at will.

Using the Laplace transform to solve PDEs

Ex: Solve
$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g, \; \partial_t u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where n is odd.

Extend u to negative times by $\tilde{u}(x,t) = \begin{cases} u(x,t), & t \geq 0 \\ u(x,-t), & t < 0 \end{cases}$

Then, $\tilde{u}_t(x,t) = -u_t(x,-t)$, $\tilde{u}_{tt}(x,t) = u_{tt}(x,-t)$. But $-t \geq 0$, where we have the wave equation, so $u_{tt}(x,-t) = \Delta u(x,-t) = \Delta \tilde{u}(x,-t)$ since spatial derivatives of $\tilde{u}(x,t)$ and $u(x,-t)$ agree. Thus

$\tilde{u}_{tt} - \Delta \tilde{u} = 0$ in $\mathbb{R}^n \times (-\infty,\infty)$ To simplify the notation, we will drop the $\sim$.

Define $\quad v(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds \quad , \quad x \in \mathbb{R}^n, \ t > 0$

$s \in \mathbb{R}$.. With some lengthy argument that will not be presented here, it can be showed that

$$\lim_{t \to 0} v(x,t) = u(x,0) = g(x).$$

Taking Δ of v: $\quad \Delta v(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} \underbrace{\Delta u(x,s)}_{= u_{ss}(x,s)}\, ds$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} u_{ss}(x,s)\, ds \underset{\text{integrate by parts}}{=} -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \frac{(-2s)}{4t} e^{-\frac{s^2}{4t}} u_s(x,s)\, ds + \overbrace{\frac{e^{-\frac{s^2}{4t}} u_s(x,s)}{\sqrt{4\pi t}} \Bigg|_{s=-\infty}^{s=+\infty}}^{=0}$$

$$\underset{\text{integrate by parts}}{=} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \left(-\frac{1}{2t} + \frac{s^2}{4t^2} \right) e^{-\frac{s^2}{4t}} u(x,s)\, ds + \overbrace{\frac{1}{\sqrt{4\pi t}} \frac{s}{2t} e^{-\frac{s^2}{4t}} u(x,s) \Bigg|_{s=-\infty}^{s=+\infty}}^{=0}$$

On the other hand: $v_t(x,t) = -\frac{1}{2(4\pi t)^{3/2}} 4\pi \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds +$

$$+ \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} \frac{s^2}{4t^2} u(x,s)\, ds = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \left(-\frac{1}{2t} + \frac{s^2}{4t^2}\right) e^{-\frac{s^2}{4t}} u(x,s)\, ds.$$

Comparing to Δv, we see that v satisfies

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

, i.e., v solves the heat equation with initial condition g. We derived its solution (p. 157):

$$v(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

Thus, using the initial definition of v:

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds = \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\frac{s^2}{4t}} u(x,s)\, ds.$$

Changing variables in the last integral: $s = -\bar{s} \Rightarrow \frac{1}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{\bar{s}^2}{4t}} \underbrace{u(x,-s)}_{= u(x,s)}\, ds$

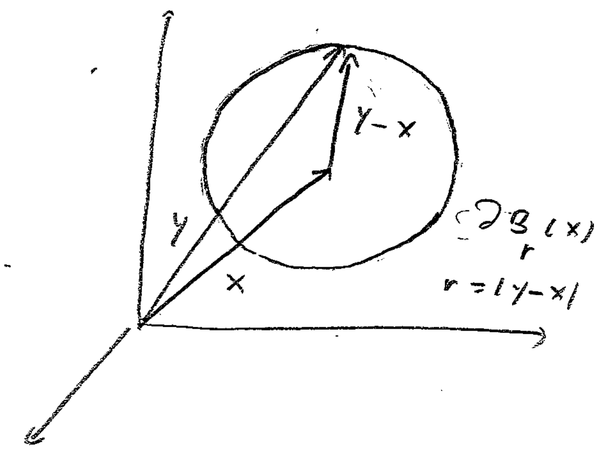
hence $v(x,t) = \frac{2}{\sqrt{4\pi t}} \int_0^{\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds$. Using also the expression for v on p. 162 and setting $\lambda = \frac{1}{4t}$ we obtain:

$$\int_0^{\infty} u(x,s)\, e^{-\lambda s^2}\, ds = \frac{1}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} \int_{\mathbb{R}^n} e^{-\lambda |x-y|^2} g(y)\, dy.$$

We are going to solve this equation for u.

Changing variables, $y - x = z$

$$\int_{\mathbb{R}^n} e^{-\lambda|x-y|^2} g(y)\, dy = \int_{\mathbb{R}^n} e^{-\lambda|z|^2} g(z+x)\, dz$$



We can integrate in "polar coordinates centered at x"

$$\int_{\mathbb{R}^n} = \int_0^\infty \int_{S^{n-1}} r^{n-1}\, dS, \quad \text{in 3d } (n=3) \int_{\mathbb{R}^n} = \int_0^\infty \underbrace{\int_0^{2\pi} \int_0^{\pi}}_{S^2 = S^{n-1} = \partial B_1(0)} \underset{r^{n-1}}{r^2}\, \underbrace{\sin\varphi\, d\varphi\, d\theta}_{dS}\, dr$$

$$\int_0^\infty \eta(x,s)\, e^{-\lambda s^2}\, ds = \frac{1}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}} n\,\alpha(n) \int_0^\infty e^{-\lambda r^2} r^{n-1} \underbrace{\frac{1}{r^{n-1} n\,\alpha(n)} \int_{\partial B_r(x)} g(y)\, \underset{dS(y)}{dS}}_{= G(x,r)}\, dr$$

where $\alpha(n)$ = volume of $B_1(0)$ in $\mathbb{R}^n$ and $n\,\alpha(n)$ = volume of $\partial B_1(0)$ in $\mathbb{R}^n$. In $n=3$: $\alpha(3) = \frac{4}{3}\pi$, $3\,\alpha(3) = 3\,\frac{4\pi}{3} = 4\pi$

$r^{n-1} n\,\alpha(n)$ = volume of $\partial B_r(0)$ (= volume of $\partial B_r(x)$). In $n=3$ $r^2 n\,\alpha(n) = 4\pi r^2$

Recall that n is odd, so $n = 2k+1$ for some k. Note that

$$-\frac{1}{2r}\frac{d}{dr}\left(e^{-\lambda r^2}\right) = \lambda e^{-\lambda r^2}$$

using successive times

$$\lambda^{\frac{n-1}{2}}\int_0^\infty e^{-\lambda r^2} r^{n-1} G(x,r)\,dr = \int_0^\infty \lambda^k e^{-\lambda r^2} r^{2k} G(x,r)\,dr = \frac{(-1)^k}{2^k}\int_0^\infty \left[\left(\frac{1}{r}\frac{d}{dr}\right)^k \left(e^{-\lambda r^2}\right)\right] \times$$

$$r^{2k} G(x,r)\,dr = \frac{1}{2^k}\int_0^\infty r\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^k \left(r^{2k-1} G(x,r)\right)\right] e^{-\lambda r^2}\,dr$$

integrate by parts k times

Thus (relabeling s by r)

$$\int_0^\infty u(x,r)\, e^{-\lambda r^2}\,dr = \frac{n\,\alpha(n)}{\pi^{\frac{n-1}{2}}\, 2^{k+1}}\int_0^\infty r\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^k \left(r^{2k-1} G(x,r)\right)\right] e^{-\lambda r^2}\,dr$$

Finally, relabel $\lambda = s$ and change variables $t = r^2$. This leads to:

$$\underbrace{\int_0^\infty u(x,t)\, e^{-st}\, dt}_{\mathcal{L}\{u\}} = \underbrace{\frac{n\,\alpha(n)}{\pi^k 2^{k+1}} \int_0^\infty \left[t \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^k \left(t^{2k-1} G(x,t) \right) \right] e^{-st}\, dt}_{\mathcal{L}\left\{ \frac{n\alpha(n)}{\pi^k 2^{k+1}} \left(t \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{k-1} \left(t^{2k-1} G(x,t)\right)\right)\right\}}$$

Like the Fourier transform, the Laplace transform can be inverted with the inverse Laplace transform (in other words, if $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f = g$). This finally gives

$$u(x,t) = \frac{1}{\gamma_n} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \frac{1}{r^{n-1} n\,\alpha(n)} \int_{\partial B_r(x)} g(y)\, dS \right), \quad \text{where } \frac{1}{\gamma_n} = \frac{n\,\alpha(n)}{\pi^k 2^{k+1}}$$

In particular, for $n = 3$ ($k = 1$), $\frac{3\,\alpha(3)}{\pi\, 4} = 1$

$$u(x,t) = \frac{\partial}{\partial t} \left(t \frac{1}{4\pi r^2} \int_{\partial B_r(x)} g(y)\, dS \right)$$

(This is an analogue of D'Alembert's formula in higher dimensions, with $\partial_t u(x,0) = 0$)

Remark The Fourier transform is useful for functions defined on $\mathbb{R}^n$, whereas the Laplace transform is useful for functions defined on $\mathbb{R}_+ = (0,\infty)$. Thus, the Fourier transform is commonly used to problems where we transform the spatial variables, while the Laplace transform is typically applied to transform the time variable.