The Schrödinger equation

we will take $\Omega = \mathbb{R}^3$. When that is not the case, one also needs boundary conditions

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta \Psi + V\Psi \quad \text{in } \Omega \subseteq \mathbb{R}^3 \ (\text{possibly } \Omega = \mathbb{R}^3)$$

where : i = complex number $i^2 = -1$

$\hbar$ = Planck's constant $= 6.63 \cdot 10^{-34}\ m^2 kg/s$

m = constant (mass)

Δ = Laplacian $= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

$x = (x_1, x_2, x_3)$

$V = V(x,t)$, $t \in \mathbb{R}$, $x \in \Omega$. $V : \Omega \times \mathbb{R} \to \mathbb{R}$ $(x = (x^1, x^2, x^3))$

$\Psi = \Psi(x,t)$, $t \in \mathbb{R}$, $x \in \Omega$, $\Psi : \Omega \times \mathbb{R} \to \mathbb{C}$ (complex function)

The Schrödinger equation describes the dynamics of a particle of mass m interacting(*) with a potential V, according to the laws of quantum mechanics. Its physical content is the following: if $\mathcal{U} \subseteq \Omega$, then $\int_{\mathcal{U}} |\Psi(x,t)|^2\, dx$ is the probability of finding the particle in the region $\mathcal{U}$ at time t.

(*) E.g.: an electron interacting with a proton

Let us assume that V does not depend on time, so $V = V(x)$ (we will eventually choose a specific V). We proceed by separation of variables so we assume that

$$\Psi(x,t) = T(t)\,\psi(x)$$

and plug into the equation to get

$$i\hbar\,\psi T' = -\frac{\hbar^2}{2m} T \Delta\psi + VT\psi \quad \text{or}$$

$$\underbrace{\frac{i\hbar T'}{T}}_{\text{depends only on } t} = \underbrace{-\frac{\hbar^2}{2m}\frac{\Delta\psi}{\psi} + V}_{\text{depends only on } x} \Longrightarrow \frac{i\hbar T'}{T} = \text{constant} = -\frac{\hbar^2}{2m}\frac{\Delta\psi}{\psi} + V$$

Call the constant E. Then
(E will have an important physical interpretation)

$$\begin{cases} -\dfrac{\hbar^2}{2m}\Delta\psi + V\psi = E\psi \\[2ex] i\hbar T' = ET \end{cases}$$

The T equation is easily solved: $\frac{T'}{T} = \frac{E}{i\hbar} \Rightarrow T(t) = T_0 e^{-\frac{iEt}{\hbar}}$ where T_0 is the initial condition for T. T_0 can be combined with further constants of integration that will appear in ψ, so we can assume $T_0 = 1$. Note that we also used that $\frac{1}{i} = -i$. Therefore:

$$\Psi(x,t) = e^{-\frac{iEt}{\hbar}} \psi(x)$$

Note that this is valid for any $V(x)$. Hence the focus in the study of Schrödinger's equation is usually on the equation for ψ:

$$-\frac{\hbar^2}{2m} \Delta\psi + V\psi = E\psi$$

which is called the <u>time independent Schrödinger equation</u>

We can write the equation as $\left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi = E\psi$. This means that the constant E is an <u>eigenvalue</u> of the <u>operator</u> $-\frac{\hbar^2}{2m}\Delta + V$

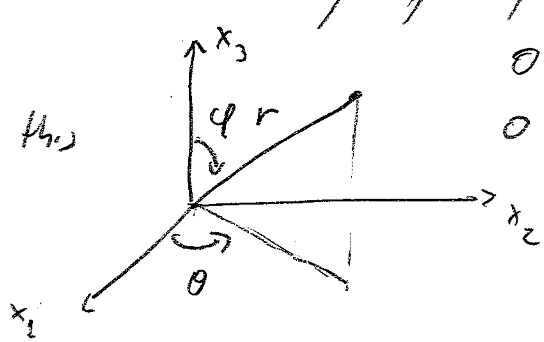
Recall from linear algebra that if A is a $n \times n$ matrix, a number λ is called an eigenvalue of A if there exists a non-zero vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. In our case it is similar, but instead of vectors in $\mathbb{R}^n$ we have functions and instead of a matrix we have an operator.

In many important physical applications V is a radially symmetric function, i.e, $V = V(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. E.g. $V(x) = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{1}{|x|} = \frac{1}{r}$. In these cases we say that V is a central potential. Let us assume that this is the case. Write

$$-\frac{\hbar^2}{2\mu} \Delta \psi + V\psi = E\psi,$$

where we relabeled m by μ for future convenience

Because $V = V(r)$, it is more appropriate to solve this problem using spherical coordinates



$0 \le r < \infty$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ $\quad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$ $\quad \varphi = \cos^{-1}\left(\frac{x_3}{r}\right)$

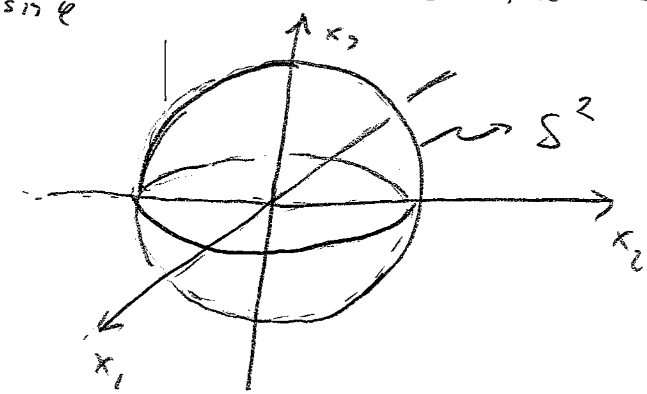
$x_1 = r\cos\theta\sin\varphi \quad x_2 = r\sin\theta\sin\varphi \quad x_3 = r\cos\varphi$

As we are working in spherical coordinates, we will use the Laplacian in spherical coordinates:

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2 \sin\varphi}\partial_\theta^2 + \frac{\cos\varphi}{r^2\sin\varphi}\partial_\varphi + \frac{1}{r^2}\partial_\varphi^2$$

$$= \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2}$$

where $\Delta_{S^2} = \partial_\varphi^2 + \frac{\cos\varphi}{\sin\varphi}\partial_\varphi + \partial_\theta^2$ is the Laplacian on the sphere $S^2 = \{(r,\theta,\varphi) \in \mathbb{R}^3 \mid r = 1\}$



We apply separation of variables to the independent Schrödinger equation as well, thus suppose

$$\psi(x) \underset{\substack{\uparrow \\ \text{simply writing in spherical coordinates}}}{=} \psi(r,\theta,\varphi) \overset{\text{separation of variables}}{=} R(r)\, Y(\theta,\varphi)$$

Plugging into the equation:

$$-\frac{\hbar^2}{2\mu}\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2}\right) R\cdot \Upsilon + VR\Upsilon = ER\Upsilon$$

$$-\frac{\hbar^2}{2\mu}\left(\partial_r^2 R + \frac{2}{r}\partial_r R\right)\Upsilon - \frac{\hbar^2}{2\mu}\frac{1}{r^2} R\,\Delta_{S^2}\Upsilon + VR\Upsilon = ER\Upsilon$$

Dividing by $\psi = R\Upsilon$ and multiplying by r^2:

$$-\frac{\hbar^2}{2\mu}\frac{r^2}{R}\left(\partial_r^2 R + \frac{2}{r}\partial_r R\right) + (V-E)r^2 = \frac{\hbar^2}{2\mu}\frac{1}{\Upsilon}\Delta_{S^2}\Upsilon$$

Since $V = V(r)$, the LHS depends only on r and the RHS only on (θ,φ). Thus

$$-\frac{\hbar^2}{2\mu}\frac{r^2}{R}\left(\partial_r^2 R + \frac{2}{r}\partial_r R\right) + (V-E)r^2 = \underbrace{-a}_{\text{constant}} = \frac{\hbar^2}{2\mu}\frac{1}{\Upsilon}\Delta_{S^2}\Upsilon$$

constant; it is convenient to write this constant as $-a$

or

$$\begin{cases} \dfrac{\hbar^2}{2\mu}\Delta_{S^2}\Upsilon = -a\Upsilon \\[2ex] -\dfrac{\hbar^2}{2\mu}\left(\partial_r^2 R + \dfrac{2}{r}\partial_r R\right) + \left(V + \dfrac{a}{r^2}\right)R = ER \end{cases}$$

Let's look first at the equation for $\overline{Y}$. Using the definition of Δ_{S^2}:

$$\partial_\varphi^2 \overline{Y} + \frac{\cos\varphi}{\sin\varphi}\partial_\varphi \overline{Y} + \frac{1}{\sin^2\varphi}\partial_\theta^2 \overline{Y} = -\frac{2\alpha p}{\hbar^2}\overline{Y}$$

Remark: note that this equation says that $-\frac{2ap}{\hbar^2}$ is an eigenvalue of Δ_{S^2}.

We again use separation of variables, thus suppose:

$$\overline{Y}(\theta,\varphi) = \Theta(\theta)\,\Phi(\varphi).$$ Plugging in:

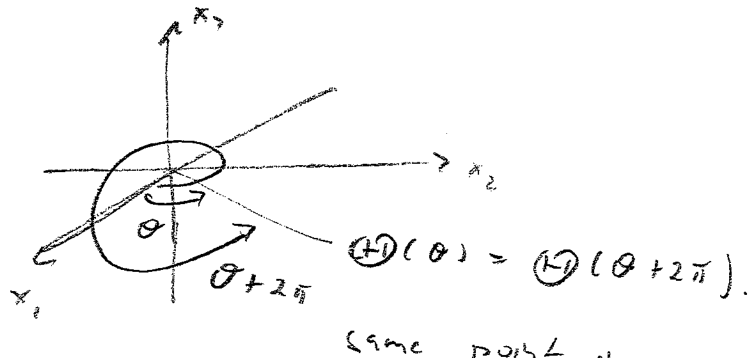
$$\Theta\Phi'' + \frac{\cos\varphi}{\sin\varphi}\Phi'\Theta + \frac{1}{\sin^2\varphi}\Phi\Theta'' = -\frac{2ap}{\hbar^2}\Theta\Phi.$$ Multiply by $\frac{\sin^2\varphi}{\Phi\Theta}$:

$$\underbrace{-\frac{\Theta''}{\Theta}}_{\text{depends only on } \theta} = \underbrace{\sin^2\varphi\frac{\Phi''}{\Phi} + \sin\varphi\cos\varphi\frac{\Phi'}{\Phi} + \frac{2\sin^2\varphi\, ap}{\hbar^2}}_{\text{depends only on } \varphi} \Longrightarrow$$ both sides are equal to a constant b

Thus $-\frac{\Theta''}{\Theta} = b$ or $\Theta'' = -b\Theta$. To solve this ODE we need boundary conditions. What are they?

Since $\Theta = \Theta(\theta)$, we should use <u>periodic boundary conditions</u>:

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$



We analyze the equation for Θ:

<u>$b=0$</u> $\Theta'' = 0 \Rightarrow \Theta(\theta) = A\theta + B$, A, B constants. The only way this can satisfy the boundary condition $\Theta(\theta + 2\pi) = \Theta(\theta)$ is if $A = 0$, thus $\Theta(\theta) = B$.

<u>$b<0$</u> $\Theta'' = -b\Theta$ gives solutions that are exponentials, which cannot be periodic. Thus $b<0$ is discarded

<u>$b>0$</u> In this case we can write $b = m^2$ for some $m \in \mathbb{R}$. Then $\Theta'' = -m^2\Theta$ gives $\Theta(\theta) = \cos(m\theta)$ or $\Theta(\theta) = \sin(m\theta)$ <u>if m is an integer</u>. If m is not an integer then the boundary condition $\Theta(\theta + 2\pi) = \Theta(\theta)$ is not satisfied. We conclude that

$$b = m^2 \quad \text{where} \quad m \in \mathbb{Z}.$$

Recalling Euler's formula $e^{i\alpha} = \cos\alpha + i\sin\alpha$, we can represent all our solutions, $\Theta(\theta) = \cos(m\theta)$, $\Theta(\theta) = \sin(m\theta)$ or $\Theta(\theta) = B$ as

$$\Theta(\theta) = e^{im\theta}, \quad m \in \mathbb{Z}.$$

Similarly to what happened with separation of variables for the wave equation, we have found infinitely many solutions to one of the ODES of the problem.

Returning to the equation for Φ, using that $b = m^2$, $m \in \mathbb{Z}$, we have

$$\sin^2\varphi \frac{\Phi''}{\Phi} + \sin\varphi\cos\varphi \frac{\Phi'}{\Phi} + \frac{2\sin^2\varphi\, a\mu}{\hbar^2} = m^2 \quad \text{or}$$

$$\sin^2\varphi \frac{\Phi''}{\Phi} + \sin\varphi\cos\varphi \frac{\Phi'}{\Phi} - m^2 = -\sin^2\varphi\,\lambda \quad \text{where} \quad \lambda = \frac{2\mu a}{\hbar^2}$$

We remark that $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$ according to $a > 0$, $a = 0$, or $a < 0$, respectively, since we don't know yet what a is (and $\frac{2\mu}{\hbar^2} > 0$).

Because $\frac{\sin\varphi}{\Phi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right) = \sin^2\frac{\Phi''}{\Phi} + \sin\varphi\cos\varphi\frac{\Phi'}{\Phi}$, we can also write

$$\frac{\sin\varphi}{\Phi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right) - m^2 = -\lambda\sin^2\varphi$$

Let us make the following change of variables: $x = \cos\varphi$. This is well-defined because $0 \leq \varphi \leq \pi$ and $\cos\varphi$ is invertible for $0 \leq \varphi \leq \pi$. Note that $-1 \leq x \leq 1$.

The chain rule gives,

$$\sin\varphi\frac{d}{d\varphi} = \sin\varphi\cdot\frac{dx}{d\varphi}\frac{d}{dx} = -\sin^2\varphi\frac{d}{dx} = -(1-\cos^2\varphi)\frac{d}{dx} = (x^2-1)\frac{d}{dx}.$$

Hence the equation reads

$$\frac{1}{\Phi}(x^2-1)\frac{d}{dx}\left((x^2-1)\frac{d\Phi}{dx}\right) - m^2 = -\lambda(1-x^2), \text{ or,}$$

$$(x^2-1)\frac{d}{dx}\left((x^2-1)\frac{d\Phi}{dx}\right) - m^2\Phi + \lambda(1-x^2)\Phi = 0 \quad \text{or equivalently}$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0$$

To solve this equation, we claim that it suffices to solve

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0,$$ and then Φ is given by $\Phi(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|}P(x)}{dx^{|m|}}$.

(Solutions Φ depends on m because the equation depends on m). To verify this claim, differentiate the equation for P:

$$(1-x^2)\frac{d^3P}{dx^3} - 4x\frac{d^2P}{dx} + (\lambda - 2)\frac{dP}{dx} = 0$$

Differentiating again:

$$(1-x^2)\frac{d^4P}{dx^4} - 6x\frac{d^3P}{dx} + (\lambda - 6)\frac{d^2P}{dx^2} = 0.$$ Proceeding this way, we see that if we differentiate the equation $|m|$ times we get:

$$(1-x^2)\frac{d^{|m|+2}P}{dx^{|m|+2}} - 2(|m|+1)x\frac{d^{|m|+1}P}{dx^{|m|+1}} + \left(\lambda - |m|(|m|+1)\right)\frac{d^{|m|}P}{dx^{|m|}} = 0$$

On the other hand, plug $\Phi(x) = (1-x^2)^{\frac{|m|}{2}}\tilde{\Phi}(x)$ into the equation for Φ, where $\tilde{\Phi}$ is a to be found function, we get:

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\left((1-x^2)^{\frac{|m|}{2}}\tilde{\Phi}\right)\right) = \frac{d}{dx}\left(\frac{|m|}{\cancel{2}}(1-x^2)^{\frac{|m|}{2}}\cdot(-\cancel{2}x)\tilde{\Phi} + (1-x^2)^{\frac{|m|}{2}+1}\frac{d\tilde{\Phi}}{dx}\right)$$

$$= -|m|(1-x^2)^{\frac{|m|}{2}}x\frac{d\tilde{\Phi}}{dx} + (1-x^2)^{\frac{|m|}{2}+1}\frac{d^2\tilde{\Phi}}{dx^2} + \left(\frac{|m|}{2}+1\right)(1-x^2)^{\frac{|m|}{2}}\cdot(-2x)\frac{d\tilde{\Phi}}{dx}$$

$$+ |m|\frac{|m|}{\cancel{2}}(1-x^2)^{\frac{|m|}{2}-1}(-\cancel{2}x)(-x)\tilde{\Phi} - |m|(1-x^2)^{\frac{|m|}{2}}\tilde{\Phi}$$

$$= (1-x^2)^{\frac{|m|}{2}+1}\frac{d^2\tilde{\Phi}}{dx^2} + (1-x^2)^{\frac{|m|}{2}}\frac{d\tilde{\Phi}}{dx}\left(\underbrace{-2x\left(\frac{|m|}{2}+1\right) - |m|x}_{= -|m|x - 2x - |m|x = -2x(|m|+1)}\right)$$

$$+ (1-x^2)^{\frac{|m|}{2}}\tilde{\Phi}\,|m|\left(\frac{|m|x^2}{1-x^2} - 1\right)$$

$$= (1-x^2)^{\frac{|m|}{2}}\left[(1-x^2)\frac{d^2\tilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\tilde{\Phi}}{dx} + |m|\left(\frac{|m|x^2}{1-x^2} - 1\right)\tilde{\Phi}\right]$$

$$= -\left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = -\left(\lambda - \frac{m^2}{1-x^2}\right)(1-x^2)^{\frac{|m|}{2}}\tilde{\Phi}$$, divide everything by $(1-x^2)^{\frac{|m|}{2}}$:

because of the equation for Φ

$$(1-x^2)\frac{d^2\tilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\tilde{\Phi}}{dx} + |m|\left(\frac{|m|x^2}{1-x^2} - 1\right)\tilde{\Phi} = -\lambda\tilde{\Phi} + \frac{m^2}{1-x^2}\tilde{\Phi}$$

We can write $m^2 = |m|^2$, so:

$$(1-x^2)\frac{d^2\tilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\tilde{\Phi}}{dx} + \underbrace{\left(\frac{|m|^2x^2}{1-x^2} - |m| - \frac{|m|^2}{1-x^2}\right)}\tilde{\Phi} + \lambda\tilde{\Phi} = 0$$

$$\frac{|m|^2x^2 - |m| + |m|x^2 - |m|^2}{1-x^2} = \frac{|m|(|m|+1)x^2 - |m|(|m|+1)}{1-x^2}$$

$$= \frac{|m|(|m|+1)(x^2-1)}{1-x^2} = -|m|(|m|+1)$$

So we finally obtained:

$$(1-x^2)\frac{d^2\tilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\tilde{\Phi}}{dx} + \left(\lambda - |m|(|m|+1)\right)\tilde{\Phi} = 0$$

Comparing with the equation for P, we see that $\tilde{\Phi} = \frac{d^{|m|}P}{dx^{|m|}}$. Thus, if we find P, we can compute $\tilde{\Phi}$, and hence Φ, what verifies the claim.

Thus, it suffices to solve

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0.$$ We seek a power series solution:

$$P(x) = \sum_{k=0}^{\infty} a_k x^k.$$ Plugging into the equation

$$(1-x^2)\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=0}^{\infty} k a_k x^{k-1} + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

$$\underbrace{\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2}} - \sum_{k=0}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} \lambda a_k x^k = 0$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k,$$ also, $-k(k-1)-2k = -k(k+1)$

$$\sum_{k=0}^{\infty}\Big(\big((k+2)(k+1)a_{k+2} - (k(k+1)-\lambda)a_k\Big)x^k = 0$$

We therefore obtain the following recurrence relation:

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, \ldots$$

This determines all coefficients a_k except a_0 and a_1, which can be prescribed arbitrarily if no further conditions are imposed (as it should be for a second order equation).

Next, we analyze the convergence of the power series, in particular the behavior at $x = \pm 1$. We remark that at this point we still haven't obtained information about λ (in particular, since Ψ is complex-valued, in principle λ could be complex)

Consider the solutions generated from a_0 and a_1 separately. The first one consists of only even powers, whereas the second only contains only odd powers. Hence, they are linearly independent.

For each of them, two consecutive coefficients satisfy

$$\left| \frac{a_{k+2}\, x^{k+2}}{a_k\, x^k} \right| \longrightarrow |x|^2 \text{ as } k \to \infty,$$ hence the series converges for $|x|<1$.

For $|x| = 1$ (ie, $x = \pm 1$), we see from the recurrence relation that

$$a_{k+2} = \frac{k^2 + \mathcal{O}(k)}{k^2 + \mathcal{O}(k)}\, a_k = \frac{k^2 + \mathcal{O}(k)}{k^2 + \mathcal{O}(k)}\, \frac{k^2 + \mathcal{O}(k)}{k^2 + \mathcal{O}(k)}\, a_{k-2} = \dots = \begin{cases} \dfrac{k^{k+2} + \mathcal{O}(k^{k+1})}{k^{k+2} + \mathcal{O}(k^{k+1})}\, a_0 & k \text{ even} \\[2ex] \dfrac{k^{k+1} + \mathcal{O}(k^{k})}{k^{k+1} + \mathcal{O}(k^{k})}\, a_1 & k \text{ odd.} \end{cases}$$

It follows that $a_k \not\to 0$ as $k \to \infty$, hence the series does not converge when $x = \pm 1$, <u>unless it reduces to a finite</u> sum, ie, unless $a_k = 0$ for all $k > l$ for some l.

Since our solution must be valid for $x = \pm 1$ (which, recall, correspond to $\varphi = 0$ and $\varphi = \pi$), we conclude that indeed $a_k = 0$ for all $k > l$.

If a_ℓ is the last non-zero term, then the recurrance relation gives

$$a_{\ell+2} = 0 = \frac{\ell(\ell+1) - \lambda}{(\ell+2)(\ell+1)} a_\ell \text{, which implies, since } a_\ell \neq 0,$$

$$\lambda = \ell(\ell+1).$$

This determines λ, and we obtain a family of solutions parametrized by ℓ, where $\ell = 0, 1, 2, \dots$ We thus denote by P_ℓ the solution with the corresponding value of ℓ. The first few P_ℓ's (after a convenient choice of a_0 and a_1 that produces integer coefficients) is

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = 1 - 3x^2, \quad P_3(x) = 3x - 5x^3.$$

We finally obtain the corresponding solution Φ by

$$\Phi(x) = (1 - x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P(x)}{dx^{|m|}}$$

Since P_ℓ is a polynomial of degree ℓ, its derivatives of order greater or equal to $\ell+1$ vanish. We conclude that, for each ℓ, the values of m that give a non-trivial solution are $|m| \le \ell$; i.e., m is restricted to be:

$$m \in \{-\ell, -\ell+1, -\ell+2, \ldots, -1, 0, 1, \ldots \ell-2, \ell-1, \ell\}.$$

We sometimes write m_ℓ when we want to stress this dependence of m on ℓ. Solutions Φ are indexed by ℓ and m, where $\ell = 0, 1, 2, \ldots$ and $|m| \le \ell$, so we write $\Phi_{\ell,m}$ or Φ_{ℓ,m_ℓ}. The first few Φ's are:

$$\Phi_{00}(x) = 1$$

$$\Phi_{1,0}(x) = x, \quad \Phi_{1,\pm 1}(x) = (1-x^2)^{1/2}$$

$$\Phi_{2,0}(x) = 1 - 3x^2, \quad \Phi_{2,\pm 1}(x) = (1-x^2)^{3/2} x, \quad \Phi_{2,\pm 2}(x) = 1 - x^2$$

$$\Phi_{3,0}(x) = 3x - 5x^3, \quad \Phi_{3,\pm 1}(x) = (1-x^2)^{3/2}(1-5x^2), \quad \Phi_{3,\pm 2}(x) = (1-x^2)\,x, \quad \Phi_{3,\pm 3}(x) = (1-x^2)^{3/2}$$

We finally return to the original variable φ, where $x = \cos\varphi$. We have

$$\Phi_{l,m}(x) = (1-x^2)^{\frac{|m|}{2}} \tilde{\Phi}_{l,m}(x), \quad \text{where } \tilde{\Phi}_{l,m} = \frac{d^{|m|} P_l}{dx^{|m|}}$$

so that

$$\Phi_{l,m}(\varphi) = \sin^{|m|}\varphi \, \tilde{\Phi}_{l,m}(\cos\varphi).$$

We now put together our results for the angular equation. Recall that $Y = \Phi \Theta$, thus we have $Y_{l,m}(\theta,\varphi) = \Theta_m(\theta)\, \Phi_{l,m}(\varphi)$, or more explicitly:

$$Y_{l,m_l}(\theta,\varphi) = e^{im_l\theta} \sin^{|m|}\varphi \, \tilde{\Phi}(\cos\varphi).$$

Recall that the equation for Y is $\frac{\hbar^2}{2\mu}\Delta_{S^2} Y = -aY$ and we saw that $\lambda = \frac{2\mu}{\hbar^2}a$, thus $a = \frac{\hbar^2}{2\mu}\lambda = \frac{\hbar^2}{2\mu}l(l+1)$, hence Y satisfies

$$\Delta_{S^2} Y = -l(l+1)\, Y \quad \left(\text{so } -l(l+1),\ l=0,1,\dots \text{ are eigenvalues of } \Delta_{S^2}\right)$$

Terminology: The equation $(1-x^2)\dfrac{d^2P_\ell}{dx^2} - 2x\dfrac{dP_\ell}{dx} + \ell(\ell+1)P_\ell = 0$

is called Legendre equation, and its solutions are called Legendre polynomials. The functions $\Phi_{l,m}$ are called associated Legendre functions

The functions $\Psi_{l,m}$ are called spherical harmonics. They are eigenfunctions (= "eigenvectors") of the operator Δ_{S^2}, and they appear in many applications that involve spherical symmetry.