VANDERBILT UNIVERSITY
MATH 294 — PARTIAL DIFFERENTIAL EQUATIONS.
HW 6.

Question 1. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Consider the problem

$$Lu + f(x,u) = 0, \text{ in } \mathbb{T}^n, \tag{1}$$

where $f \in C^\infty(\mathbb{T}^n \times \mathbb{R})$, and

$$Lu = a^{ij}\partial_{ij}u + b^i\partial_i u + cu$$

is an elliptic operator (notice the difference in sign convention as compared to the textbook). Assume the coefficients of L are smooth. Notice that problem (1) is in general non-linear (take, for instance, $f(x,u) = u^2$), and that a boundary condition is not prescribed since $\mathbb{T}^n$ is boundaryless.

We say that $u_- \in C^2(\mathbb{T}^n)$ is a sub-solution of (1) if $Lu_- + f(x,u_-) \geq 0$. Analogously, we say that $u_+ \in C^2(\mathbb{T}^n)$ is a super-solution of (1) if $Lu_+ + f(x,u_+) \leq 0$. The goal of this problem is to show that if we can find suitable sub- and super-solutions, then problem (1) has in fact a (smooth) solution.

Assume from now one there there exist u_-, u_+, sub- and super-solutions of (1) such that

$$u_- \leq u_+.$$

(a) Let A be a constant such that $-A \leq u_- \leq u_+ \leq A$. Using the compactness of $\mathbb{T}^n$, show that there exists a large positive number γ such that

$$F(x,t) = \gamma t + f(x,t)$$

is increasing in $t \in [-A, A]$ for any fixed $x \in \mathbb{T}^n$.

(b) Define

$$Pu = -Lu + \gamma u.$$

Show that γ can be chosen large enough such that, as an operator from $C^{2,\alpha}(\mathbb{T}^n)$ to $C^\alpha(\mathbb{T}^n)$, P has a compact inverse P^{-1}.

(c) Using the maximum principle, show that P is a positive operator, i.e., if $Pv_1 \geq Pv_2$ then $v_1 \geq v_2$.

(d) Define inductively:

$$\phi_0 = u_-,\ \phi_k = P^{-1}(F(x,\phi_{k-1})),$$

and,

$$\psi_0 = u_+,\ \psi_k = P^{-1}(F(x,\psi_{k-1})).$$

Show that

$$P\phi_0 \leq P\phi_1 = F(x,\phi_0) \leq F(x,\psi_0) = P\psi_1 \leq P\psi_0.$$

(e) Conclude that

$$\phi_0 \leq \phi_1 \leq \psi_1 \leq \psi_0.$$

(f) Prove inductively that we obtain in this way sequences $\{\phi_k\}$ and $\{\psi_k\}$ such that

$$\phi_0 \leq \phi_{k-1} \leq \phi_k \leq \psi_k \leq \psi_{k-1} \leq \psi_0.$$

(f) Using the monotonicity and boundedness of the above sequences, conclude that they converge poin-twise to limits $\phi_k \to \Phi$ and $\psi_k \to \Psi$ satisfying $\phi_0 \leq \Phi \leq \Psi \leq \psi_0$.

(g) Combine the definition and boundedness of ϕ_k and ψ_k with L^p estimates for a linear equation to conclude that the sequences $\{\phi_k\}$ and $\{\psi_k\}$ are bounded in $W^{2,p}(\mathbb{T})$. Invoke the Sobolev embedding theorem to conclude that the point-wise convergence found above is in fact $C^{1,\alpha}$ convergence.

(h) Invoke elliptic regularity and Schauder estimates to obtain that the convergence is in fact $C^{2,\alpha}$ convergence. Use this to conclude that

$$Pv = F(x, v), \tag{2}$$

where $v = \Phi$ or Ψ.

(i) Conclude that (2) holds if and only if (1) does. Invoke elliptic regularity to obtain a smooth solution of (1).

(j) State the result obtained above as a theorem. Can you lower the regularity of the coefficients of L and obtain the same result?