VANDERBILT UNIVERSITY
MATH 294 — PARTIAL DIFFERENTIAL EQUATIONS.
HW 3.

Question 1. Solve $uu_x + u_y = 1$ with $u = 0$ when $y = x$. Something bad happens if we replace the condition $u = 0$ by $u = 1$.

Question 2. Show that $xu_y - yu_x = x^2 + y^2$ has no continuous solution in any neighborhood of $(0, 0)$. (Hint: write the equation in polar coordinates).

Question 3. Let

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0, \end{cases}$$

be the Heaviside function. Show that

$$H' = \delta,$$

where δ is the Dirac delta distribution.

Question 4. Let

$$L = \sum_{j=0}^{2} c_j \frac{d^j}{dx^j}$$

be an ordinary differential operator with constant coefficients and $c_2 \neq 0$. Let v be a solution of $Lv = 0$ satisfying $v(0) = 0$, $v'(0) = c_2^{-1}$. Define

$$f(x) = \begin{cases} 0, & x \leq 0, \\ v(x), & x > 0. \end{cases}$$

Show that

$$Lf = \delta$$

where δ is the Dirac delta distribution.

Question 5 (optional). Prove a generalized Liouville theorem: If u is harmonic on $\mathbb{R}^n$ and $|u(x)| \leq C(1 + |x|)^N$ for some $C, N > 0$, then u is a polynomial. (Hint: the estimate on u implies that u is a tempered distribution so it has a Fourier transform. You may also use, without proof, that the only distribution whose support is $\{0\}$ are the linear combinations of the Dirac delta at zero and its derivatives).

Question 6. Explain what is meant by the following statement: solutions of the heat equation exhibit infinite propagation speed, whereas those of the wave equation exhibit finite propagation speed.

Question 7. Let u solve

$$\begin{cases} \Box u = 0, & \text{in } (0,\infty) \times \mathbb{R}^3, \\ u = g, & \text{on } \{t=0\} \times \mathbb{R}^3, \\ u_t = h, & \text{on } \{t=0\} \times \mathbb{R}^3, \end{cases}$$

where g and h are smooth and have compact support. Show that there exists a constant C such that

$$|u(t,x)| \leq \frac{C}{t},$$

for $x \in \mathbb{R}^3$ and $t > 0$.