MATH 2610 - TEST 3 SOLUTIONS

VANDERBILT UNIVERSITY

NAME:

Directions:

- Unless stated otherwise, the notation and conventions used in class apply to this test.
- Provide full justifications for your answer. Answers without justification will receive little or no credit.
- Write clearly and legibly.

Question	Points	Score
1	10	
2	20	
3	10	
4	20	
5	20	
6	20	
TOTAL	100	

List of formulas

Below are formulas you are allowed to use. Note that it is not said for which kind of equation or in which context each formula applies. You need to recognize them from class and the homework.

If

$$(x, y) = c_1 u + c_2 v,$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are linearly independent vectors, then

$$c_1 = \frac{v_2 x_0 - v_1 y_0}{u_1 v_2 - v_1 u_2},\ c_2 = \frac{-u_2 x_0 + u_1 y_0}{u_1 v_2 - v_1 u_2}.$$

Question 1. (10 points) The questions that follow refer to the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y).$$

(a) (2 points) What is a critical point for this system and how is it related to solutions of the system?

(b) (2 points) What is an isolated critical point?

(c) (3 points) Define what it means to say that a critical point is stable, asymptotically stable, and unstable. Illustrate the definitions with pictures.

(d) (3 points) Define an almost linear system near the origin.

Solution 1. (a) A point (x_0, y_0) is a critical point if it satisfies $f(x_0, y_0) = 0 = g(x_0, y_0)$. In this case, the constant functions $x(t) = x_0$, $y(t) = y_0$ are a solution to the system.

(b) A critical point (x_0, y_0) is isolated if there exists a neighborhood D of (x_0, y_0) such that (x_0, y_0) is the only critical point in D.

(c) A critical point (x_0, y_0) is stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $(x(t),\, y(t))$ satisfying

$$\sqrt{(x(0) - x_0)^2 + (y(0) - y_0)^2} < \delta$$

also satisfies

$$\sqrt{(x(t) - x_0)^2 + (y(t) - y_0)^2} < \varepsilon$$

for all $t \geq 0$. If (x_0, y_0) is stable and there exists a $\eta > 0$ such that any solution $(x(t),\, y(t))$ satisfying

$$\sqrt{(x(0) - x_0)^2 + (y(0) - y_0)^2} < \eta$$

converges to (x_0, y_0) as $t \to \infty$, then the critical point is called asymptotically stable. If a critical point is not stable then it is called unstable.

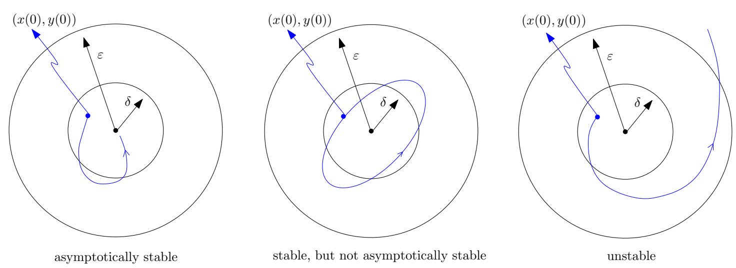


FIGURE 1. Illustration of stability/instability.

(d) Consider the system

$$\dot{x} = ax + by + F(x, y),$$
$$\dot{y} = cx + dy + G(x, y),$$

where a, b, c, d are constant, F and G are continuous in a neighborhood of the origin, and assume that the origin is a critical point. Suppose that $ad - bc \neq 0$. The system is almost linear near the origin if

$$\frac{F(x, y)}{\sqrt{x^2 + y^2}} \to 0 \text{ and } \frac{G(x, y)}{\sqrt{x^2 + y^2}} \to 0 \text{ as } (x, y) \to (0, 0).$$

Question 2. (20 points) Consider the linear system

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

and suppose that $ad - bc \neq 0$. Let λ_1 and λ_2 be the eigenvalues of the system. Based on the definition of stability/instability you gave in question 1, show that:

(a) (10 points) The system is asymptotically stable if $\lambda_1, \lambda_2 < 0$, $\lambda_1 \neq \lambda_2$.

(b) (10 points) The system is stable if the eigenvalues are purely imaginary, i.e., $\lambda_1 = i\beta$, $\lambda_2 = -i\beta$, $\beta \in \mathbb{R}$, $\beta \neq 0$.

Solution 2. (a) Solutions take the form

$$(x(t), y(t)) = c_1 e^{\lambda_1 t} u + c_2 e^{\lambda_2 t} v,$$

where u and v are linearly independent eigenvectors. Writing c_1, c_2 in terms of $(x_0, y_0) = (x(0), y(0))$, we find

$$(x(t), y(t)) = \frac{v_2 x_0 - v_1 y_0}{u_1 v_2 - v_1 u_2} e^{\lambda_1 t} u + \frac{-u_2 x_0 + u_1 y_0}{u_1 v_2 - v_1 u_2} e^{\lambda_2 t} v.$$

We have

$$\|(x(t), y(t))\| \leq \frac{4}{|u_1 v_2 - v_1 u_2|} \|v\| \|u\| e^{-\min\{|\lambda_1|, |\lambda_2|\} t} \|(x_0, y_0)\|.$$

Thus, given $\varepsilon > 0$, we have $\|(x(t), y(t))\| < \varepsilon$ for $t \geq 0$ whenever $\|(x_0, y_0)\| < \delta$, with

$$\frac{4}{|u_1 v_2 - v_1 u_2|} \|v\| \|u\| \delta < \varepsilon.$$

Asymptotic stability then follows from $e^{-\min\{|\lambda_1|, |\lambda_2|\} t} \to 0$ as $t \to \infty$.

(b) Solutions take the form

$$(x(t), y(t)) = c_1(\cos(\beta t) u - \sin(\beta t) v) + c_2(\sin(\beta t) u + \cos(\beta t) v),$$

where u and v are linearly independent. Writing c_1, c_2 in terms of $(x_0, y_0) = (x(0), y(0))$, we find

$$(x(t), y(t)) = \frac{v_2 x_0 - v_1 y_0}{u_1 v_2 - v_1 u_2} (\cos(\beta t) u - \sin(\beta t) v) + \frac{-u_2 x_0 + u_1 y_0}{u_1 v_2 - v_1 u_2} (\sin(\beta t) u + \cos(\beta t) v).$$

Then

$$\begin{aligned} \|(x(t), y(t))\| &\leq \frac{2}{|u_1 v_2 - v_1 u_2|} \|v\| \|(x_0, y_0)\| (\|u\| + \|v\|) + \frac{2}{|u_1 v_2 - v_1 u_2|} \|u\| \|(x_0, y_0)\| (\|u\| + \|v\|) \\ &\leq \frac{2}{|u_1 v_2 - v_1 u_2|} \|(x_0, y_0)\| (\|u\| + \|v\|)^2, \end{aligned}$$

where we used that $|\cos(\beta t)| \leq 1$ and $|\sin(\beta t)| \leq 1$. Thus, given $\varepsilon > 0$, we have $\|(x(t), y(t))\| < \varepsilon$ for $t \geq 0$ whenever $\|(x_0, y_0)\| < \delta$, with

$$\frac{2}{|u_1 v_2 - v_1 u_2|} (\|u\| + \|v\|)^2 \delta < \varepsilon.$$

Question 3. (10 points) Show that the system

$$\dot{x} = \sin(y - 3x)$$
$$\dot{y} = \cos x - e^y,$$

is almost linear near the origin and discuss its stability.

Solution 3. With $f(x, y) = \sin(y-3x)$ and $g(x, y) = \cos x - e^y$, we find $f_x(0, 0) = -3$, $f_y(0, 0) = 1$, $g_x(0, 0) = 0$, and $g_y(0, 0) = -1$, and write the system as

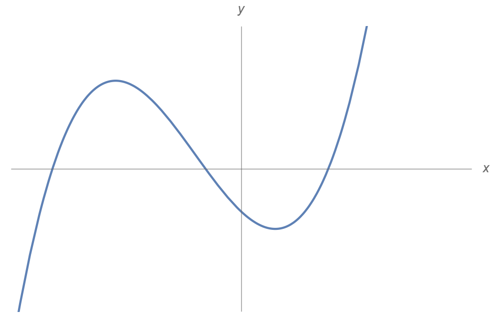
$$\dot{x} = -3x + y + F(x, y),$$
$$\dot{y} = -y + G(x, y),$$

with $F(x, y) = 3x - y + \sin(y - 3x)$ and $G(x, y) = y + \cos x - e^y$. We have

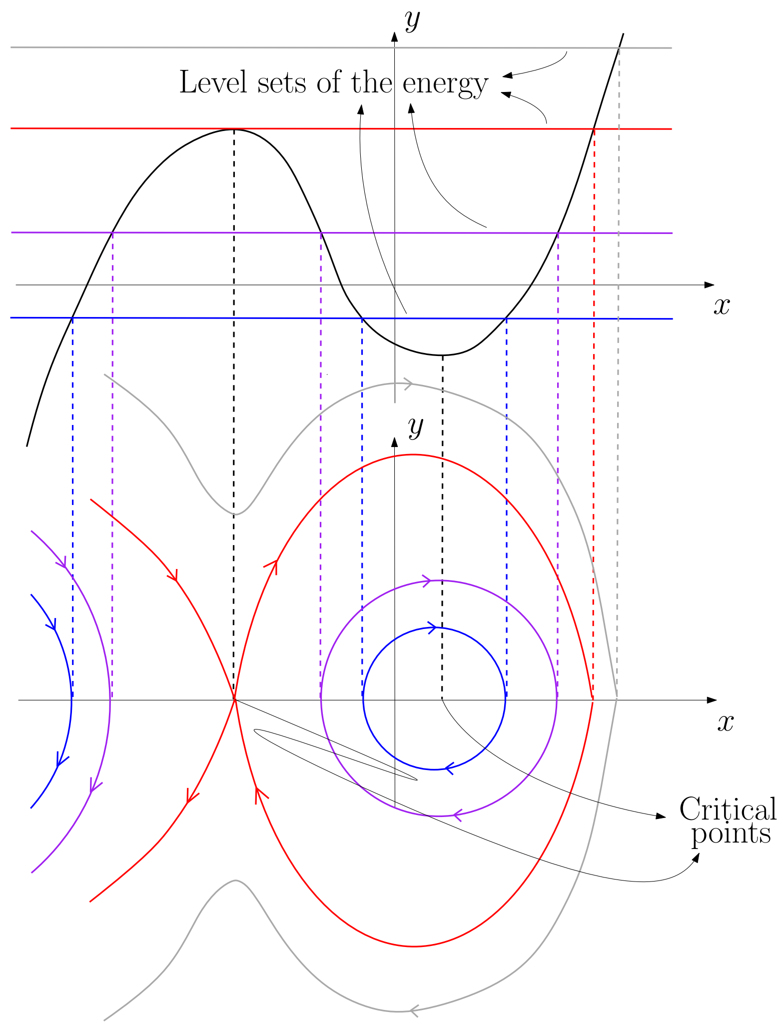
$$\begin{aligned} F(x, y) &= 3x - y + \sin(y - 3x) = 3x - y + y - 3x + O((y - 3x)^3) \\ &= O(x^2 + y^2), \\ G(x, y) &= y + \cos x - e^y = y + 1 - \frac{x^2}{2!} + O(x^4) - (1 + y + \frac{y^2}{2!} + O(y^3)) \\ &= O(x^2 + y^2) \end{aligned}$$

which gives $\frac{F(x,y)}{\|(x,y)\|} \to 0$ and $\frac{G(x,y)}{\|(x,y)\|} \to 0$ as $(x, y) \to 0$, showing that the system is almost linear. The linear part of the system has eigenvalues -1 and -3, thus the origin is asymptotically stable.

Question 4. (20 points) Consider a conservative system whose potential function is given by the graph below. Sketch the phase portrait of the system.



Solution 4. The solution is:



Question 5. (20 points) Consider the conservative system

$$\ddot{x} + g(x) = 0.$$

Assume that g is continuous, that $g(0) = 0$, $g(x) \neq 0$ for $x \neq 0$, and that there exists a constant $k > 0$ such that $xg(x) < 0$ for $0 < |x| < k$. Show that the origin is an unstable critical point for this system. *Hint:* You do not need to do $\epsilon - \delta$ arguments here. Rather, consider the potential function and use the given information to understand its behavior in a neighborhood of $x = 0$.

Solution 5. The potential function is $G(x) = \int g(x)\, dx + C$, where C is a constant chosen to set the zero of the energy function. The critical points are given by $(x_0, 0)$ where $G'(x_0) = g(x_0) = 0$, thus the origin is the only critical point. $xg(x) < 0$ for $0 < |x| < k$ means that, for x near 0, $g(x) > 0$ for $x < 0$ and $g(x) < 0$ for $x > 0$. Thus, for x near 0, $G'(x) > 0$ for $x < 0$ and $G'(x) < 0$ for $x > 0$. Therefore, 0 is a local maximum of G. This means that, qualitatively, near the origin trajectories behave like the red and gray curves in the solution of the previous question, giving that the origin is unstable.

Question 6. Prove that the equation

$$\ddot{x} + (x^4 + (\dot{x})^2 - 1)\dot{x} + x = 0$$

has a non-constant periodic solution.

Solution 6. Write the system as

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -(x^4 + y^2 - 1)y - x.\end{aligned}$$

We see that $(0,0)$ is the only critical point of this system. Consider $V(x,y) = ax^m + by^n$. Then

$$\begin{aligned}\frac{d}{dt}V(x,y) &= amx^{m-1}\dot{x} + bny^{n-1}\dot{y}\\ &= amx^{m-1}y - bny^{n-1}((x^4 + y^2 - 1)y + x).\end{aligned}$$

If we choose $m = n = 2$ and $a = b = 1$ we find

$$\frac{d}{dt}V(x,y) = 2xy - 2y((x^4 + y^2 - 1)y + x) = -2y^2(x^4 + y^2 - 1) = y^2(1 - (x^4 + y^2)).$$

Consider the curve γ given by $x^4 + y^2 = 1$. Then, $\frac{d}{dt}V(x(t), y(t))$ is ≥ 0 inside γ and ≤ 0 outside γ. The curve γ lies between the circle $x^2 + y^2 = 1$ and the square $\{(x,y) \in \mathbb{R}^2 \mid \max(|x|, |y|) = 1\}$, touching them at the points $(\pm 1, 0)$ and $(0, \pm 1)$. We can choose as the region R of the Poincaré-Bendixson theorem any annulus $r_A \leq x^2 + y^2 \leq r_B$ with $0 < r_A < 1$ and $r_B > \sqrt{2}$. Applying this theorem then gives the result.