MATH 2610 - PRACTICE FOR TEST 3, SOLUTIONS

VANDERBILT UNIVERSITY

Question 1. The questions that follow refer to the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y).$$

(a) What is a critical point for this system and how is it related to solutions of the system?

(b) What is an isolated critical point?

(c) Define what it means to say that a critical point is stable, asymptotically stable, and unstable. Illustrate the definitions with pictures.

(d) Define an almost linear system near the origin.

Solution 1. (a) A point (x_0, y_0) is a critical point if it satisfies $f(x_0, y_0) = 0 = g(x_0, y_0)$. In this case, the constant functions $x(t) = x_0$, $y(t) = y_0$ are a solution to the system.

(b) A critical point (x_0, y_0) is isolated if there exists a neighborhood D of (x_0, y_0) such that (x_0, y_0) is the only critical point in D.

(c) A critical point (x_0, y_0) is stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $(x(t),\, y(t))$ satisfying

$$\sqrt{(x(0) - x_0)^2 + (y(0) - y_0)^2} < \delta$$

also satisfies

$$\sqrt{(x(t) - x_0)^2 + (y(t) - y_0)^2} < \varepsilon$$

for all $t \geq 0$. If (x_0, y_0) is stable and there exists a $\eta > 0$ such that any solution $(x(t),\, y(t))$ satisfying

$$\sqrt{(x(0) - x_0)^2 + (y(0) - y_0)^2} < \eta$$

converges to (x_0, y_0) as $t \to \infty$, then the critical point is called asymptotically stable. If a critical point is not stable then it is called unstable.

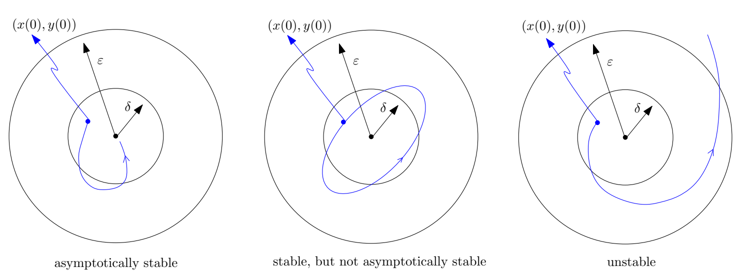


FIGURE 1. Illustration of stability/instability.

(d) Consider the system

$$\dot{x} = ax + by + F(x, y),$$

$$\dot{y} = cx + dy + G(x, y),$$

where a, b, c, d are constant, F and G are continuous in a neighborhood of the origin, and assume that the origin is a critical point. Suppose that $ad - bc \neq 0$. The system is almost linear near the origin if

$$\frac{F(x,y)}{\sqrt{x^2+y^2}} \to 0 \text{ and } \frac{G(x,y)}{\sqrt{x^2+y^2}} \to 0$$

as $(x, y) \to (0, 0)$.

Question 2. Consider the linear system

$$\dot{x} = ax + by,$$
$$\dot{y} = cx + dy,$$

and suppose that $ad - bc \neq 0$. Let λ_1 and λ_2 be the eigenvalues of the system. Based on the definition of stability/instability you gave in question 1, show that:

(a) The system is asymptotically stable if $\lambda_1, \lambda_2 < 0$.

(b) The system is unstable if one of the eigenvalues is positive.

Solution 2. (a) Since $\lambda_1, \lambda_2 < 0$, solutions take the form

$$(x(t), y(t)) = c_1 e^{\lambda_1 t} u + c_2 e^{\lambda_2 t} v,$$

where u and v are linearly independent eigenvectors, or

$$(x(t), y(t)) = (c_1 u + c_2 v + c_2 t w) e^{\lambda t},$$

where u is an eigenvector, v a generalized eigenvetor linearly independent from u, $w = (A - \lambda I)v$, and $\lambda_1 = \lambda_2 = \lambda$. Writing c_1, c_2 in terms of $(x_0, y_0) = (x(0), y(0))$, we find

$$(x(t), y(t)) = \frac{v_2 x_0 - v_1 y_0}{u_1 v_2 - v_1 u_2} e^{\lambda_1 t} u + \frac{-u_2 x_0 + u_1 y_0}{u_1 v_2 - v_1 u_2} e^{\lambda_2 t} v,$$

or

$$(x(t), y(t)) = (\frac{v_2 x_0 - v_1 y_0}{u_1 v_2 - v_1 u_2} u + \frac{-u_2 x_0 + u_1 y_0}{u_1 v_2 - v_1 u_2} v + \frac{-u_2 x_0 + u_1 y_0}{u_1 v_2 - v_1 u_2} t w) e^{\lambda t}.$$

In the first case, we have

$$\|(x(t), y(t))\| \leq \frac{4}{|u_1 v_2 - v_1 u_2|} \|v\| \|u\| e^{-\min\{|\lambda_1|, |\lambda_2|\} t} \|(x_0, y_0)\|.$$

Thus, given $\varepsilon > 0$, we have $\|(x(t), y(t))\| < \varepsilon$ for $t \geq 0$ whenever $\|(x_0, y_0)\| < \delta$, with

$$\frac{4}{|u_1 v_2 - v_1 u_2|} \|v\| \|u\| \delta < \varepsilon.$$

Asymptotic stability then follows from $e^{-\min\{|\lambda_1|, |\lambda_2|\} t} \to 0$ as $t \to \infty$.

In the second case, we have

$$\begin{aligned} \|(x(t), y(t))\| &\leq \frac{1}{|u_1 v_2 - v_1 u_2|} (4\|v\| \|u\| + 2\|u\| \|w\| t) e^{-|\lambda| t} \|(x_0, y_0)\| \\ &\leq \frac{1}{|u_1 v_2 - v_1 u_2|} (4\|v\| \|u\| + 2\|u\| \|w\| \frac{1}{|\lambda| e}) \|(x_0, y_0)\|, \end{aligned}$$

where in the second step we used that $te^{-|\lambda| t}$ has a maximum at $t = \frac{1}{|\lambda|}$ and $t \geq 0$. Thus, given $\varepsilon > 0$, we have $\|(x(t), y(t))\| < \varepsilon$ for $t \geq 0$ whenever $\|(x_0, y_0)\| < \delta$, with

$$\frac{1}{|u_1 v_2 - v_1 u_2|} (4\|v\| \|u\| + 2\|u\| \|w\| \frac{1}{|\lambda| e}) \delta < \varepsilon.$$

Asymptotic stability then follows from $e^{-|\lambda|t} \to 0$ and $te^{-|\lambda|t} \to 0$ as $t \to \infty$.

(b) Let $\lambda_1 > 0$. Then

$$(x(t), y(t)) = c_1 e^{\lambda_1 t} u$$

is a solution. Since $\|e^{\lambda_1 t} u\| \to \infty$ as $t \to \infty$, if we take $\varepsilon = 1$, then for every $\delta > 0$, if $\|(x_0, y_0)\| = |c_1| \|u\| < \delta$, we can find $t_0 > 0$ such that $\|(x(t_0), y(t_0))\| = |c_1| e^{\lambda_1 t_0} \|u\| > 1$, showing that the solution is unstable.

Question 3. Show that the system

$$\begin{aligned} \dot{x} &= e^{x+y} - \cos x, \\ \dot{y} &= \cos y + x - 1, \end{aligned}$$

is almost linear near the origin and discuss its stability.

Solution 3. Write, using Taylor's series for the exponential and cosine,

$$\begin{aligned} \dot{x} &= e^{x+y} - \cos x = 1 + x + y + \frac{(x+y)^2}{2!} + O((x+y)^3) - (1 - \frac{x^2}{2!} + O(x^4)) \\ &= x + y + O(x^2 + y^2), \\ \dot{y} &= \cos y + x - 1 = 1 + \frac{y^2}{2!} + O(y^4) + x - 1 \\ &= x + O(x^2 + y^2). \end{aligned}$$

(Alternatively, we can use the linearization of the system to obtain the same result.) Thus the system can be written as

$$\begin{aligned} \dot{x} &= x + y + F(x, y), \\ \dot{y} &= x + 0y + G(x, y), \end{aligned}$$

with $F(x, y), G(x, y) = O(x^2 + y^2)$ so that $\frac{|F(x,y)|}{\|(x,y)\|}, \frac{|G(x,y)|}{\|(x,y)\|} \to 0$ as $(x, y) \to (0, 0)$. Since $ad - bc \neq 0$, the system is almost linear. The eigenvalues of the linear part are $\frac{1 \pm \sqrt{5}}{2}$, which gives an unstable critical point.

Question 4. Consider the DE

$$\ddot{x} + e^x - 1 = 0.$$

(a) Explain why this is a conservative system.

(b) Find the potential function G.

(c) Find the energy function $E(x, v)$. Select it so that $E(0, 0) = 0$.

(d) Write the DE as a first order system and determine its critical points.

(e) Determine the stability of the critical points.

Solution 4. (a) The system can be written as $\ddot{x} = F(x)$, thus this is a conservative system.

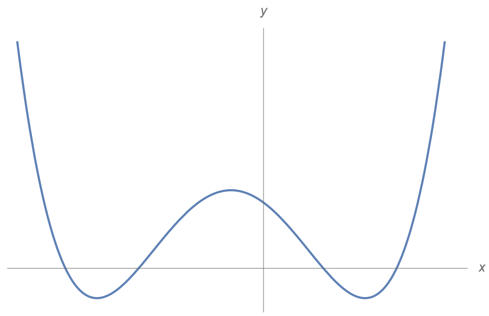
(b) We write $\ddot{x} + g(x) = 0$, with $g(x) = e^x - 1$. Then $G(x) = \int g(x)\, dx = e^x - x + C$.

(c) The energy function is given by $E(x, v) = \frac{1}{2} v^2 + G(x)$. Choosing $C = -1$ we have $E(x, v) = \frac{1}{2} v^2 + e^x - x - 1$, so $E(0, 0) = 0$.

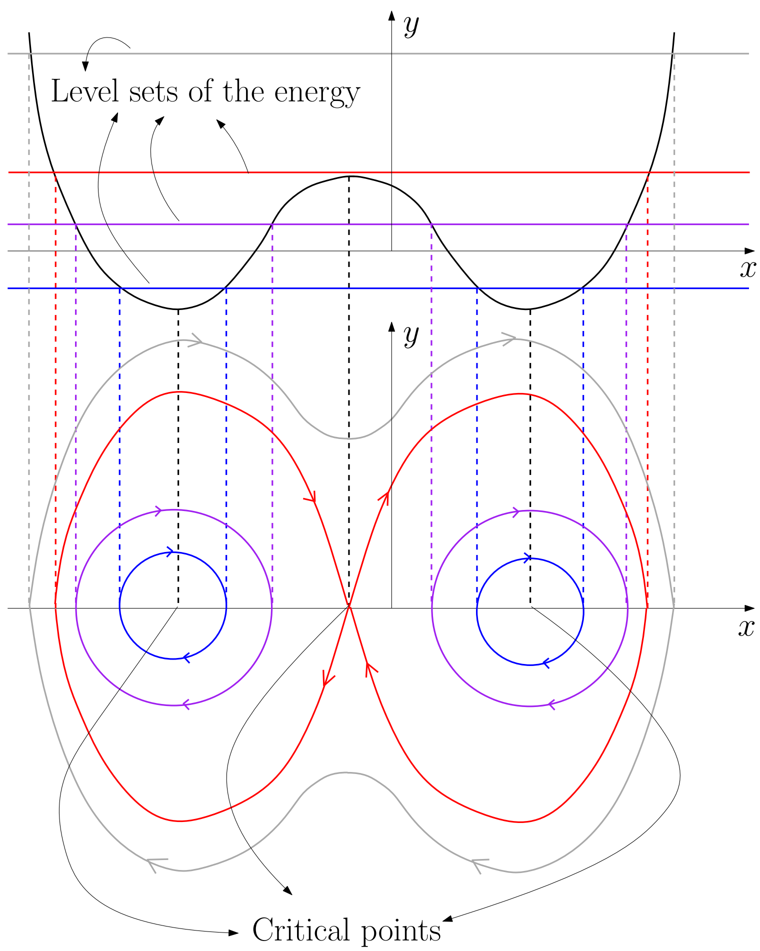
(d) We have $\dot{x} = v$, $\dot{v} = -e^x + 1$.

(e) The only critical point is $(0, 0)$. Since $G'(0) = g(0) = 0$ and $G''(0) = 1 > 0$, $x = 0$ is a local minimum of G and we conclude that the critical point is a center.

Question 5. Consider a conservative system whose potential function is given by the graph below. Sketch the phase portrait of the system.



Solution 5. The solution is:



Question 6. Consider the system

$$\dot{x} = x^3y + x^2y^3 - x^5,$$
$$\dot{y} = -2x^4 - 6x^3y^2 - 2y^5.$$

(a) Show that the origin is a critical point.

(b) Explain why this system is not almost linear.

(c) Determine the stability of the origin. To do so, you can use, without proving it, that the origin is an isolated critical point. *Hint:* the function $ax^2 + by^2$ is useful.

Solution 6. (a) This follows by plugging $(0,0)$ into the right hand side and observing that the resulting expressions vanish.

(b) The corresponding linear part has $a = b = c = d = 0$ thus it fails the condition $ad - bc \neq 0$.

(c) Let $V(x, y) = ax^2 + by^2$ and compute

$$\begin{aligned}\frac{d}{dt}V(x,y) &= V_x(x,y)\dot{x} + V_y(x,y)\dot{y} \\ &= 2ax(x^3y + x^2y^3 - x^5) + 2by(-2x^4 - 6x^3y^2 - 2y^5) \\ &= 2a(x^4y + x^3y^3 - x^6) + 2b(-2x^4y - 6x^3y^3 - 2y^6).\end{aligned}$$

If we choose $a = 2$, $b = 1$, we get

$$\frac{d}{dt}V(x,y) = -8x^3y^3 - 4x^6 - 4y^6 = -4(x^3 + y^3)^2,$$

which is negative semi-definite. By the Lyapunov stability theorem the critical point is stable.

Question 7. Determine the stability of the origin for the system

$$\begin{aligned}\dot{x} &= 2x^3, \\ \dot{y} &= 2x^2y - y^3.\end{aligned}$$

Hint: the function $x^2 - y^2$ is useful.

Solution 7. We note that the origin is an isolated critical point. Compute

$$\begin{aligned}W(x,y) &= V_x(x,y)f(x,y) + V_y(x,y)g(x,y) \\ &= 4x^4 - 4x^2y^2 + 2y^4 \\ &= 2x^4 + 2(x^2 - y^2)^2.\end{aligned}$$

We see that W is positive definite. Since $V(0,0) = 0$ and for any disk centered at the origin we can find $(x_0, 0)$ with $V(x_0, 0) > 0$, the Lyapunov instability theorem gives that the origin is unstable.

Question 8. Prove that the equation

$$\ddot{x} + (x^4 + (\dot{x})^2 - 1)\dot{x} + x = 0$$

has a non-constant periodic solution.

Solution 8. Write the system as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -(x^4 + y^2 - 1)y - x.\end{aligned}$$

We see that $(0,0)$ is the only critical point of this system. Consider $V(x,y) = ax^m + by^n$. Then

$$\begin{aligned}\frac{d}{dt}V(x,y) &= amx^{m-1}\dot{x} + bny^{n-1}\dot{y} \\ &= amx^{m-1}y - bny^{n-1}((x^4 + y^2 - 1)y + x).\end{aligned}$$

If we choose $m = n = 2$ and $a = b = 1$ we find

$$\frac{d}{dt}V(x,y) = 2xy - 2y((x^4 + y^2 - 1)y + x) = -2y^2(x^4 + y^2 - 1) = y^2(1 - (x^4 + y^2)).$$

Consider the curve γ given by $x^4 + y^2 = 1$. Then, $\frac{d}{dt}V(x(t), y(t))$ is ≥ 0 inside γ and ≤ 0 outside γ. The curve γ lies between the circle $x^2 + y^2 = 1$ and the square $\{(x,y) \in \mathbb{R}^2 \mid \max(|x|, |y|) = 1\}$, touching them at the points $(\pm 1, 0)$ and $(0, \pm 1)$. We can choose as the region R of the Poincaré-Bendixson theorem any annulus $r_A \leq x^2 + y^2 \leq r_B$ with $0 < r_A < 1$ and $r_B > \sqrt{2}$. Applying this theorem then gives the result.