VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Test 1 — Solutions

NAME: Solutions.

Directions. This exam contains six questions and an extra credit question. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you need to use a theorem that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (10 pts)	
2 (20 pts)	
3 (10 pts)	
4 (25 pts)	
5 (15 pts)	
6 (20 pts)	
Extra Credit (05 pts)	
TOTAL (100 pts)	

Question 1. (10 pts) For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $y\dfrac{dy}{dx} + \dfrac{y}{x} = 0.$

(b) $x'''' + \cos t\, x' = \sin t.$

(c) $y''' = -\cos y\, y'.$

Solution 1. (a) Unknown function: y; non-linear; first order. (b) Unknown function: x; linear; fourth order. (c) Unknown function: y; non-linear; third order.

Question 2. (20 pts)
(a) State a theorem that assures that the initial value problem

$$x' = f(t, x),$$
$$x(t_0) = x_0,$$

admits a unique solution in a neighborhood of t_0.
(b) Using the theorem you stated in (a), show that

$$3x' - t^2 + tx^3 = 0,$$
$$x(1) = 6,$$

has a unique solution defined in a neighborhood of $t = 1$.
(c) Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Can you use the theorem you stated in (a) to guarantee that the initial value problem

$$x' = f(x),$$
$$x(0) = 0,$$

admits a unique solution?

Solution 2. (a) If f and $\partial_x f$ are continuous on a rectangle $R \subseteq \mathbb{R}^2$ containing the point (t_0, x_0), then the initial value problem $x' = f(t, x)$, $x(t_0) = x_0$, admits a unique solution defined in some neighborhood of t_0.

(b) We have $f(t, x) = t^2/3 - tx^3/3$, with $\partial_x f(t, x) = -tx^2$. Both functions are continuous in a neighborhood of $(1, 6)$, thus the theorem stated in (a) applies.

(c) The given function is continuous and differentiable, but $\partial_x f$ is not continuous at $x = 0$. Therefore, the theorem stated in (a) cannot be applied.

Question 3. (10 pts) Solve the following initial value problems.

(a) $y' = \dfrac{y-1}{x+3}$, $y(-1) = 0$.

(b) $x' = e^{-t} - 4x$, $x(0) = \dfrac{4}{3}$.

Solution 3. (a) This equation is separable, so

$$\frac{dy}{y-1} = \frac{dx}{x+3} \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3},$$

for $y \neq 1$, from what we obtain

$$|y-1| = C|x+3|,$$

or yet

$$y = 1 + C(x+3).$$

Using the initial condition we find $C = -\frac{1}{2}$, thus

$$y = 1 - \frac{1}{2}(x+3).$$

(b) This is a linear equation with $p(t) = 4$ and $q(t) = e^{-t}$. Using the formula for first order linear equations, we find

$$e^{\int p(t)\,dt} = e^{4t},$$

so that

$$x = \frac{1}{3}e^{-t} + Ce^{-4t}.$$

The initial condition gives $C = 1$, so

$$x = \frac{1}{3}e^{-t} + e^{-4t}.$$

Question 4. (25 pts) Solve the following differential equations.

(a) $t^2x'' + 8tx' + 16x = 0$, $t > 0$.

(b) $y' = \dfrac{\cos y \cos x + 2x}{\sin y \sin x + 2y}$.

(c) $x^2y' = y - 1$, $x > 0$.

(d) $x'' - 2x' + x = \dfrac{e^t}{t}$, $t > 0$.

Solution 4. (a) This is a Cauchy-Euler equations with $a = 1$, $b = 8$, $c = 16$. The characteristic equation becomes $\lambda^2 + (8-1)\lambda + 16 = 0$, so $\lambda = (-7 \pm \sqrt{15}\, i)/2$, giving $x(t) = c_1 t^{-7/2}\cos(\sqrt{15}\ln(t)/2) + c_2 t^{-7/2}\cos(\sqrt{15}\ln(t)/2)$, where c_1 and c_2 are arbitrary constants.

(b) Write the equation as $(\cos y \cos x + 2x)dx - (\sin y \sin x + 2y)dy = 0$. We readily verify that this equation is exact, with $M(x, y) = \cos y \cos x + 2x$ and $N(x, y) = -(\sin y \sin x + 2y)$. Then

$$F(x, y) = \int M(x, y)\, dx = \sin x \cos y + x^2 + g(y).$$

From $\frac{\partial F}{\partial y} = N$, we find $g'(y) = -2y$, hence $g(y) = -y^2$. The general solution is

$$F(x, y) = \sin x \cos y + x^2 - y^2 = C.$$

(c) This is a separable equation:

$$\frac{dy}{y-1} = \frac{dx}{x^2},\ y \neq 1.$$

Integrating, we get

$$\ln|y - 1| = -\frac{1}{x} + C,$$

which leads to

$$y = Ce^{-x^{-1}} + 1.$$

The solution $y = 1$ is included in the above family upon taking $C = 0$.

(d) The characteristic equation is $\lambda^2 - 2\lambda + 1 = 0$, so $x_1(t) = e^t$ and $x_2(t) = te^t$ are two linearly independent solutions of the associated homogeneous equation. Next, we use the variation of parameters formula

$$x_p(t) = -\frac{x_1}{a}\int \frac{f(t)x_2(t)}{W(x_1, x_2)(t)}\, dt + \frac{x_2}{a}\int \frac{f(t)x_1(t)}{W(x_1, x_2)(t)}\, dt.$$

Computing each term, we find $W(e^t, te^t) = e^{2t}$, $\int \frac{e^t}{t}\frac{te^t}{e^{2t}}\, dt = t$, $\int \frac{e^t}{t}\frac{e^t}{e^{2t}}\, dt = \ln|t| = \ln t$. Thus

$$x_p(t) = -te^t + te^t \ln t,$$

and $x(t) = c_1e^t + c_2te^t - te^t + te^t \ln t$ is the general solution, where c_1 and c_2 are arbitrary constants.

Question 5. (15) Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.

(a) $x'' + 2x' - 3x = e^t + t^3$.

(b) $x'' + 4x = \sin(4t)$.

Solution 5. (a) The characteristic equation of the associated homogeneous equation is $\lambda^2 + 2\lambda - 3 = 0$ giving $x_h(t) = c_1 e^t + c_2 e^{-3t}$. Using the superposition principle and the method of undetermined coefficients we find

$$x_p(t) = Ate^t + Bt^3 + Ct^2 + Dt + E,$$

where $A, \dots, E$ are constant to be determined.

(b) The characteristic equation of the associated homogeneous equation is $\lambda^2 + 4 = 0$ giving $x_h(t) = c_1 \cos(2t) + c_2 \sin(2t)$. Using the method of undetermined coefficients we find

$$x_p(t) = A\cos(4t) + B\sin(4t),$$

where A and B are constant to be determined.

Question 6. (20 pts) Find the general solution of

$$(1-t)x'' + tx' - x = 0,\ 0 < t < 1.$$

Hint: e^t is a solution to the differential equation.

Solution 6. We will use the formula for a second linearly independent solution with $x_1(t) = e^t$ and $p(t) = \frac{t}{1-t}$ (note that $p(t) \neq t$):

$$x_2(t) = e^t \int \frac{e^{-\int \frac{t}{1-t}\,dt}}{e^{2t}}\,dt = e^t \int \frac{e^{t+\ln(1-t)}}{e^{2t}}\,dt = e^t \int e^{-t}(1-t)\,dt = e^t e^{-t} t = t.$$

So the general solution is

$$x(t) = c_1 e^t + c_2 t,$$

where c_1 and c_2 are arbitrary constants.

Extra credit. (05 pts) State and prove the superposition principle for second order linear differential equations (not necessarily with constant coefficients).

Solution 7. Done in class, see the class notes.