VANDERBILT UNIVERSITY

MATH 2420 – METHODS OF ORDINARY DIFFERENTIAL EQUATIONS

Examples of section 8.6

Question. Solve

$$x^2y'' + xy' + (x^2-4)y = 0.$$

Solution. Write the equation as

$$y'' + \frac{1}{x}y' + \frac{x^2-4}{x^2}y = 0.$$

Then $x=0$ is a singular point, since

$$p(x) = \frac{1}{x},$$

and

$$q(x) = \frac{x^2-4}{x^2}.$$

This singular point is a regular singular point because

$$xp(x) = 1,$$

and

$$x^2q(x) = x^2 - 4,$$

which are polynomials, thus analytic functions.

Computing

$$p_0 = \lim_{x\to 0} xp(x) = 1,$$

and

$$q_0 = \lim_{x\to 0} x^2q(x) = -4,$$

we find the indicial equation to be

$$r(r-1) + p_0 r + q_0 = r^2 - 4 = 0.$$

The solutions of the indicial equation are $r=2$ and $r=-2$. As discussed in class, when there are two distinct roots, we ought to take the larger one. Thus, we put $r=2$, and look for a solutions of the form

$$y = x^2\sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty a_n x^{n+2}.$$

Differentiating and plugging into the equation yields

$$\sum_{n=0}^\infty (n+2)(n+1)a_n x^{n+2} + \sum_{n=0}^\infty (n+2)a_n x^{n+2} + \sum_{n=0}^\infty a_n x^{n+4} - 4\sum_{n=0}^\infty a_n x^{n+2} = 0.$$

After some algebra, this becomes

$$5a_1x^3 + \sum_{n=2}^{\infty}((n^2+4n)a_n + a_{n-2})x^{n+2} = 0.$$

This gives

$$a_1 = 0,$$

and

$$a_n = -\frac{1}{n(n+4)}a_{n-2}.$$

Then,

$$a_2 = -\frac{1}{2(6)}a_0 = -\frac{1}{2^1(3!)}a_0,$$

$$a_3 = 0,$$

$$a_4 = -\frac{1}{4(8)}a_2 = \frac{1}{2^3(2!)(4!)}a_0,$$

$$a_5 = 0,$$

$$a_6 = -\frac{1}{6(10)}a_4 = -\frac{1}{2^5(3!)(5!)}a_0,$$

$$a_7 = 0,$$

and we see that

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n-1}n!(n+2)!},$$

and

$$a_{2n+1} = 0.$$

We find as solution

$$y = x^2\sum_{n=0}^{\infty}\frac{(-1)^n a_0}{2^{2n-1}n!(n+2)!}x^{2n}.$$