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Math 234

Derivation of the 1D Heat Equation

Consider a cylindrical rod of uniform cross sectional area (A_c). This rod is insulated on its lateral surface, which means there is only heat transfer along the major axis of the rod. Designate a coordinate system at one end with at x=0 and denote the entire length as L. Let the temperature function, u, be dependent upon the position and time (e.g., *u=u(x,t)*). Now we look at a differential element along this rod with length Δx, from x_0 to x_0+Δx.

Newton's Law of Cooling states that the quantity of heat flux, Q", flowing across a point x_0 per unit time is proportional to the temperature gradient at x_0:

$$Q''(x_0) \propto \frac{du(x_0,t)}{dx} = u_x(x_0,t)$$

We can add a proportionality constant and we get:

$$Q''(x_0) = cu_x(x_0,t)$$

The unit of heat flux, Q" is Watt/m^2. If the cross-sectional area (A_c) is constant, we can expand this to:

$$Q(x_0) = -kA_c u_x(x_0,t)$$

In the above formula, Q is given in units of Watts (W). Recall that 1 W = 1 Joule/sec, where Joule is a measure of energy, work, or heat. We often put a negative sign for convention's sake, and we denote *k* as the thermal conductivity of a material. Materials with higher thermal conductivities provide low resistances to heat transfer and as a result have higher heat fluxes across them (i.e. metals). Thermal conductors have high thermal conductivities and thermal insulators have low thermal conductivities.

Now we look at the heat loss from x_0 to x_0+Δx per unit time:

$$H = Q_{x_0} - Q_{x_0+\Delta x} = kA_c[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]$$

If we were to conduct this experiment, we would find that the average change in temperature Δu is proportional to the total quantity of heat introduced H*Δt, and inversely proportional to the mass of the object. The proportionality constant is, by convention, given as 1/s, where s is the specific heat of the material.

We therefore, have:

$$\Delta u = \frac{H\Delta t}{sm}$$

where s is the specific heat of the substance. Specific heat is essentially the energy required to increase the temperature of a unit mass of a substance by 1°C. Now substituting this into the former equation, we get:

$$\frac{\Delta u}{\Delta t} = \frac{kA_c[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]}{sm}$$

We know:

m = density * volume = density * area * length

Mathematically, we can write this as : $m = \rho V = \rho A_c \Delta x$

And so we get:

$$\frac{\Delta u}{\Delta t} = \frac{kA_c[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]}{s(\rho A_c \Delta x)}$$

$$\frac{\Delta u}{\Delta t} = \left(\frac{k}{\rho s}\right)\frac{[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]}{\Delta x}$$

Taking the limit as Δx → 0 and Δt → 0, we get:

$$U_t = \left(\frac{k}{\rho s}\right)U_{xx}$$

The term ($\frac{k}{\rho s}$) is often called the thermal diffusivity of a material and is denoted as c^2. We also note that k, s, and rho are all material properties and are all positive in value. Therefore, we now have the 1D heat equation:

$$U_t = c^2 U_{xx}$$

Note that this is the homogeneous form of the equation; if there is heat added or removed from the rod, the equation becomes $U_t = c^2 U_{xx} + f(x,t)$. The function f(x,t) represents the heat source or sink density. We have seen problems where we have to solve the 1D heat equation with some prescribed conditions.

For instance, we can say that the ends of the rod, x=0 and x=L are kept at temperatures T_0 and T_1, respectively and that the initial temperature distribution of the rod is f(x). Now this translates to boundary conditions of $u(0,t)=T_0$ and $u(L,t)=T_1$, while the initial condition is u(x,0) = f(x). We have seen how to solve this by using separation of variables.

The derivations are similar in 2D and 3D, and the forms of homogeneous equations are similar as well:

$$2D: U_t = c^2(U_{xx} + U_{yy})$$
$$3D: U_t = c^2(U_{xx} + U_{yy} + U_{zz})$$