MATH 2300-04
TEST 1 – SOLUTIONS

VANDERBILT UNIVERSITY

Directions: This exam contains seven questions and an extra credit question. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The notation and conventions in this exam are the same as used in class, unless stated otherwise.

If you need to use a theorem or formula that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems or formulas you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (8 pts)	
2 (12 pts)	
3 (15 pts)	
4 (15 pts)	
5 (15 pts)	
6 (15 pts)	
7 (20 pts)	
Extra Credit (5 pts)	
TOTAL (100 pts)	

Question 1. (8 pts) let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^3$ and c, d be scalars. For each expression below, identify whether or not it is well-defined.

(a) $\dfrac{\mathbf{u}}{\mathbf{v}} + c$.

(b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

(c) $c\mathbf{u} - d\mathbf{v} + \mathbf{u} \times \mathbf{v}$.

(d) $(c\mathbf{u}) \cdot (d\mathbf{v}) + cd$.

Solution 1. (a) Undefined, the quotient of two vectors is not a well-defined operation. (b) Well-defined, the cross product of two vectors is a vector, the scalar product takes two vectors as arguments. (c) Well-defined, a scalar times a vector is a vector, the cross product of two vectors is a vector, sum and subtraction of two vectors is a vector. (d) Well-defined, a scalar times a vector is a vector, the dot product of two vectors is a scalar.

Question 2. (12 pts) Sketch the regions in $\mathbf{R}^3$ determined by the given equations.

(a) $x^2 + y^2 + z^2 = 9$ and $z^2 - x^2 - y^2 > 0$.

(b) $z > x^2 + y^2$ and $z < -x^2 - y^2 + 1$.

Solution 2. (a) $x^2 + y^2 + z^2 = 9$ is a sphere of radius 3 and $z^2 - x^2 - y^2 > 0$ is the region in the interior of the cone $z^2 = x^2 + y^2$.

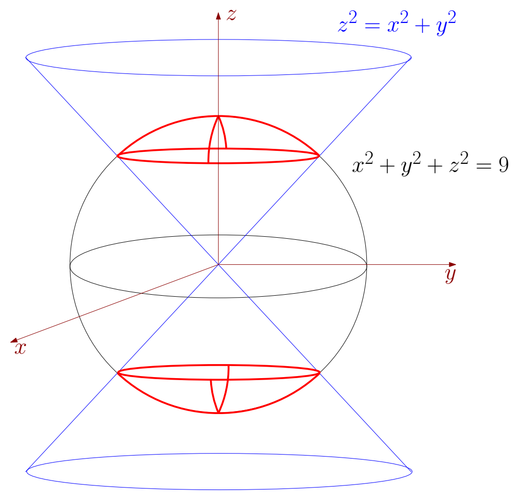

FIGURE 1. Problem 2a.

(b) $z > x^2+y^2$ is the region in the interior of the paraboloid $z = x^2+y^2$ and $z < -x^2-y^2+1$ is the region in the interior of the paraboloid $z = -x^2 - y^2 + 1$.

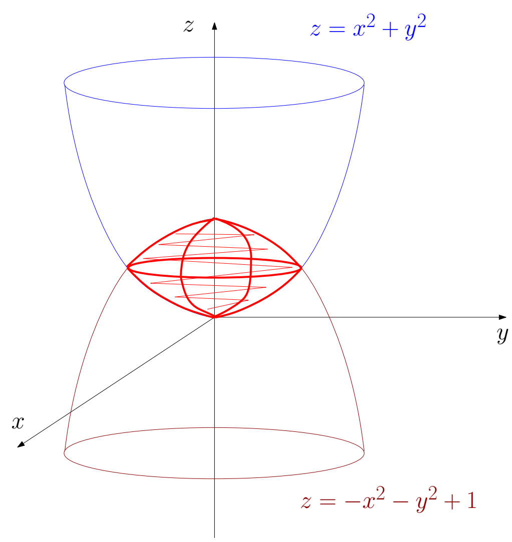

FIGURE 2. Problem 2b.

Question 3. (15 pts) Write an equation for the plane that is tangent to the sphere $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$.

Hint: Draw a picture, and use it to identify a vector normal to the plane.

Solution 3. A normal to the plane is $\vec{k} = (0, 0, 1)$ (see figure 3). Thus an equation for the plane is

$$(0, 0, 1) \cdot ((x, y, z) - (0, 0, 1)) = 0,$$

or $z = 1$.

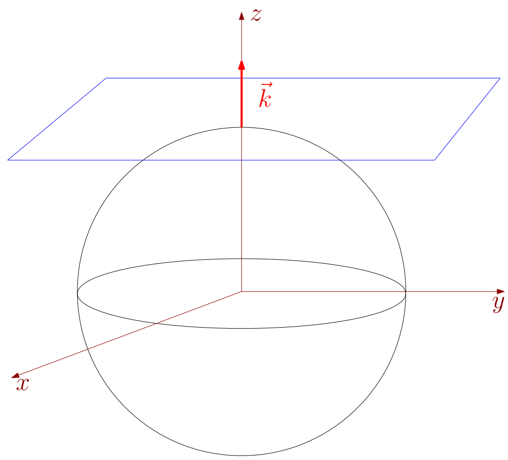

FIGURE 3. Problem 3.

Question 4. (15 pts) For each vector-valued function below, find its domain and points of discontinuity (if any).

(a) $\mathbf{r}(t) = \ln(t-1)\,\mathbf{i} + \dfrac{t}{t-2}\,\mathbf{j} + \sqrt{10-t}\,\mathbf{k}$.

(b) $\mathbf{r}(t) = \mathbf{v}(t) \times \mathbf{u}(t)$, where $\mathbf{v}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{u}(t) = f(t)\,\mathbf{i} + e^t\,\mathbf{j} + \dfrac{1}{f(t)}\,\mathbf{k}$,

where $f(t)$ is given by

$$f(t) = \begin{cases} t, & t \neq 1, \\ 2, & t = 1. \end{cases}$$

Solution 4. (a) We have

$$\begin{aligned} \ln(t-1) &\Rightarrow t > 1, \\ \frac{t}{t-2} &\Rightarrow t \neq 2, \\ \sqrt{10-t} &\Rightarrow t \leq 10. \end{aligned}$$

Thus, the domain is $(1,2) \cup (2,10]$, and $\mathbf{r}$ is continuous on its domain.

(b) $f(0) = 0$, thus $\mathbf{u}$ is not defined for $t = 0$ because of the term $\frac{1}{f(t)}$. $\mathbf{v}$ and $\mathbf{u}$ are defined for all other values of t, thus the domain of $\mathbf{r}$ is $\{t \neq 0\}$. $\mathbf{v}$ is continuous and $\mathbf{u}$ are continuous on its domain except for $t = 1$ since f is not continuous there. Thus, $\mathbf{r}$ is continuous on its domain except for $t = 1$.

Question 5. (15 pts)

(a) State the definition of curvature of a smooth curve.

(b) Show that the curvature of a circle of radius R is $1/R$, where $R > 0$.

Solution 5. (a) Consider a smooth curve. Let $\mathbf{r}$ be a vector-valued function representing the curve and assume that $\mathbf{r}$ is parametrized by arc-length. The curvature is defined as

$$\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|,$$

where $\mathbf{T}$ is the unit tangent vector and s the arc length parameter.

(b) We can represent a circle of radius R by $\mathbf{r}(t) = (R\cos t, R\sin t, 0)$, $R > 0$. Then $\mathbf{r}'(t) = (-R\sin t, R\cos t, 0)$, $\mathbf{r}''(t) = (-R\cos t, -R\sin t, 0)$. Compute

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin t & R\cos t & 0 \\ -R\cos t & -R\sin t & 0 \end{bmatrix} = R^2\mathbf{k}.$$

Then

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{R^2}{(R^2\sin^2 t + R^2\cos^2 t)^{\frac{3}{2}}} = \frac{1}{R}.$$

Question 6. (15 pts) Let $\mathbf{r}(t) = (2t, 2t, 2t)$. Write $\mathbf{r}$ parametrized in terms of arc length.

Solution 6. The arc length is given by

$$s(t) = \int_0^t |\mathbf{r}'(\tau)| \, d\tau = \int_0^t \sqrt{2^2 + 2^2 + 2^2} \, d\tau = 2\sqrt{3}t.$$

Therefore,

$$t = \frac{s}{2\sqrt{3}}.$$

Plugging into the expression for $\mathbf{r}$ we find

$$\mathbf{r}(s) = (\frac{1}{\sqrt{3}}s, \frac{1}{\sqrt{3}}s, \frac{1}{\sqrt{3}}s).$$

Question 7. (20 pts) Let $\mathbf{r}$ be a smooth curve defined on the interval $[-1, 1]$. The following information is known about $\mathbf{r}$:

- $\mathbf{r}(-1) = (1, 1, 1)$,
- $\mathbf{r}(1) = (3, -2, 1)$,
- The arc length function corresponding to $\mathbf{r}$ is $s(t) = e^t$.
- The curve $\mathbf{r}$ has no self-intersections, i.e., there do not exist $t_0, t_1 \in [-1, 1]$ such that $\mathbf{r}(t_0) = \mathbf{r}(t_1)$ and $t_0 \neq t_1$.

Find the length of the curve traced by $\mathbf{r}(t)$ from $(1, 1, 1)$ to $(3, -2, 1)$.

Hint: Use the most basic property of the arc length. Do not overthink the properties stated above; they are there simply to help you justify why such basic property of $s(t)$ can be invoked.

Solution 7. Since there are no self intersections, $\mathbf{r}(t) = (1, 1, 1) \Leftrightarrow t = -1$ and $\mathbf{r}(t) = (3, -2, 1) \Leftrightarrow t = 1$. From the properties of the arc length, therefore, the length of the curve traced by $\mathbf{r}(t)$ from $(1, 1, 1)$ to $(3, -2, 1)$ is

$$L = s(1) - s(-1) = e - e^{-1}.$$

Extra credit (5 pts). Let $\mathbf{r}$ and f be a differentiable vector-valued function and a differentiable scalar function, respectively. Assume that $\mathbf{r}$ is defined on the interval (c, d), and that f is defined on the interval (a, b). Assume that $f(t) \in (c, d)$ for every $t \in (a, b)$, so that $\mathbf{r}(f(t))$ is well-defined for $t \in (a, b)$. Show that

$$(\mathbf{r}(f(t)))' = \mathbf{r}'(f(t))\, f'(t),$$

$t \in (a, b)$.

Solution 8. Writing $\mathbf{r} = (r_1, r_2, r_3)$ and using the chain rule for single-variable functions, we have

$$\begin{aligned}
(\mathbf{r}(f(t)))' &= (r_1(f(t)), r_2(f(t)), r_3(f(t)))' \\
&= ((r_1(f(t)))', (r_2(f(t)))', (r_3(f(t)))') \\
&= (r_1'(f(t))f'(t), r_2'(f(t))f'(t), r_3'(f(t))f'(t)) \\
&= (r_1'(f(t)), r_2'(f(t))r_3'(f(t)))\, f'(t) \\
&= \mathbf{r}'(f(t))f'(t).
\end{aligned}$$