MATH 2300-04
PRACTICE TEST 1 - SOLUTIONS

VANDERBILT UNIVERSITY

Directions: This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the first test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The first test will cover the following sections of the textbook: 12.1, 12.2, 12.3, 12.4, 12.5, 12.6, 13.1, 13.2, and 13.3.

Question 1. let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^3$ and c, d be scalars. For each expression below, identify whether or not it is well-defined.

(a) $\mathbf{u} \cdot \mathbf{v} + c$.

(b) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$.

(c) $\mathbf{u} \times \mathbf{v} + c$.

(d) $(c\mathbf{u}) \times (d\mathbf{v}) + (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$.

Solution 1. (a) Well-defined: $\mathbf{u} \cdot \mathbf{v}$ is a scalar and so is c. (b) Ill-defined $\mathbf{v} \cdot \mathbf{w}$ is a scalar and one cannot take the dot product of a vector $\mathbf{u}$ with a scalar. (c) Ill-defined: $\mathbf{u} \times \mathbf{v}$ is a vector and c a scalar, and one cannot add a vector to a scalar. (d) Well-defined: $c\mathbf{u}$ is a vector, $d\mathbf{v}$ is a vector, thus $(c\mathbf{u}) \times (d\mathbf{v})$ is a vector; $\mathbf{v} \cdot \mathbf{w}$ is a scalar, thus $(\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ is a vector.

Question 2. Sketch the following surfaces in $\mathbf{R}^3$.

(a) $x^2 + 2z^2 = 1$.

(b) $x^2 + y^2 + z^2 = 4$ and $z > \sqrt{x^2 + y^2}$.

Solution 2. (a) $x^2 + 2z^2 = 1$ is an ellipse in the xz-plane, so the surface is a cylinder, see figure 1.

(b) $x^2 + y^2 + z^2 = 4$ is a sphere of radius 2 and $z > \sqrt{x^2 + y^2}$ is the interior of the cone $z = \sqrt{x^2 + y^2}$. The intersection is the sphere cap belonging to the interior of the cone, see figure 2.

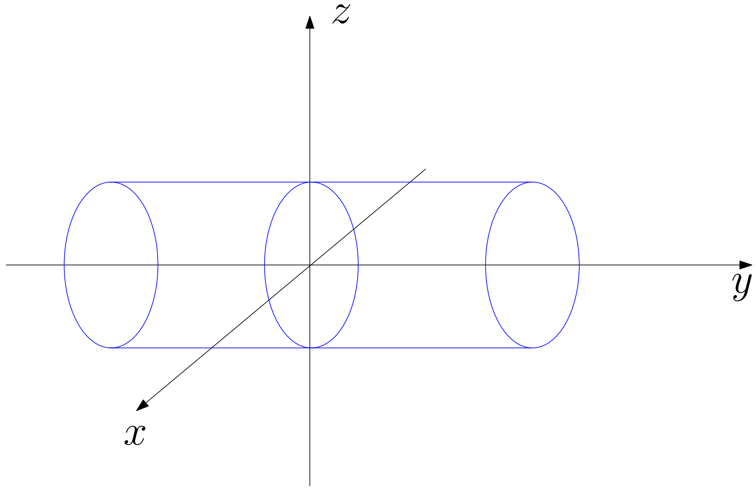

FIGURE 1. Problem 2a.

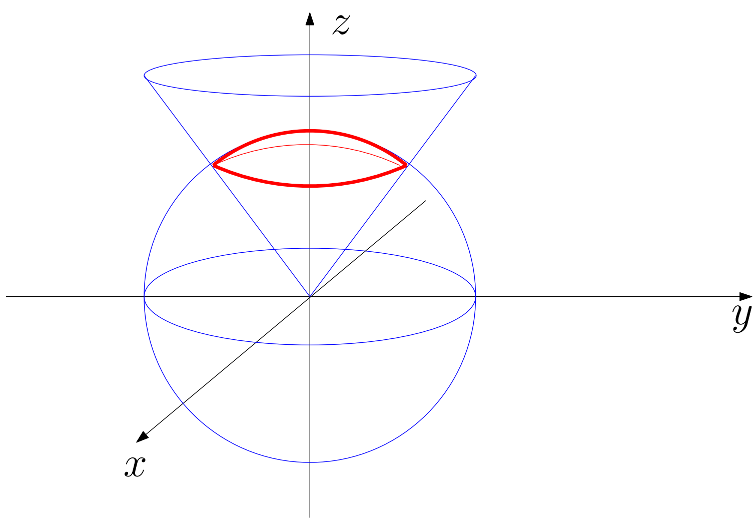

FIGURE 2. Problem 2b.

Question 3. Consider the planes $x + 2y - z + 3 = 0$ and $x + by - z + b + 1 = 0$. Find the values of b for which both planes intersect orthogonally, and then find the equation of the line determine by the intersection.

Solution 3. The first plane has normal $\vec{n}_1 = (1, 2, -1)$ and the second plane has normal $\vec{n}_2 = (1, b, -1)$. For the planes to be orthogonal we need $\vec{n}_1 \cdot \vec{n}_2 = 0$, thus $(1, 2, -1) \cdot (1, b, -1) = 1 + 2b + 1 = 0$, thus $b = -1$.

To determine the line given by the intersection, first we find the direction of the line by

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} = -3\vec{i} - 3\vec{k}.$$

Next, we need a point on the line, which is determined by finding a point belonging to both planes, i.e., satisfying the system

$$\begin{cases} x + 2y - z + 3 &= 0, \\ x - y - z &= 0. \end{cases}$$

Setting $z = 0$ we have

$$\begin{cases} x + 2y + 3 &= 0, \\ x - y &= 0. \end{cases}$$

Solving, we find $x = -1$ and $y = -1$, thus $(-1, -1, 0)$ belongs to the line, which is given by $\vec{r}(t) = t\vec{v} + (-1, -1, 0)$.

Question 4. For each vector-valued function below, find its domain and points of discontinuity (if any).

(a) $\mathbf{r}(t) = (t^2, 1, (t+1)^{-2})$.

(b) $\mathbf{r}(t) = \mathbf{v}(t) \times \mathbf{u}(t)$, where $\mathbf{v}(t) = e^{-t^2}\mathbf{i} + \cos t\mathbf{j} + t^3\mathbf{k}$, $\mathbf{u}(t) = t\mathbf{i} + \sin(\cos t)\mathbf{j} + (t+2)^4\mathbf{k}$.

(a) $\mathbf{r}(t) = (\ln t, \dfrac{1}{t-5}, \sqrt{t-2})$.

Solution 4. (a) The argument of $(t+1)^{-2}$ cannot vanish, thus $D = \{t \neq -1\}$. (b) Both $\mathbf{v}$ and $\mathbf{u}$ have domain $\mathbb{R}$, so does $\mathbf{r}$. (c) We have:

$$\ln t \Rightarrow t > 0,$$
$$\frac{1}{t-5} \Rightarrow t \neq 5,$$
$$\sqrt{t-2} \Rightarrow t \geq 2,$$

so $D = [2, 5) \cup (5, \infty)$.

Question 5. Let $\mathbf{r}$ and f be a differentiable vector-valued function and a differentiable scalar function, respectively, both defined on an interval (a, b). Show that

$$(f(t)\mathbf{r}(t))' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t),$$

$t \in (a, b)$.

Solution 5. Writing $\mathbf{r}(t) = (x(t), y(t), z(t))$, we have

$$\begin{aligned} (f(t)\mathbf{r}(t))' &= ((f(t)x(t))', (f(t)y(t))', (f(t)z(t))') \\ &= (f'(t)x(t) + f(t)x'(t), f'(t)y(t) + f(t)y'(t), f'(t)z(t) + f(t)z'(t)) \\ &= f'(t)(x(t), y(t), z(t)) + f(t)(x'(t), y'(t), z'(t)) \\ &= f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t). \end{aligned}$$

Question 6. What is the curvature of an ellipse?

Solution 6. Write $\mathbf{r}(t) = (a\cos t, b\sin t, 0)$, $a, b > 0$. Then $\mathbf{r}'(t) = (-a\sin t, b\cos t, 0)$, $\mathbf{r}''(t) = (-a\cos t, -b\sin t, 0)$. Compute

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin t & b\cos t & 0\\ -a\cos t & -b\sin t & 0\end{bmatrix} = ab\mathbf{k}.$$

Then

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}.$$

Question 7. Let $\mathbf{r}(t) = (\frac{t^2}{2}, \sin t, \cos t)$. Find the length of the curve traced by $\mathbf{r}(t)$ from $t = 0$ to $t = \frac{\pi}{4}$ in two different ways: (a) by using the formula for the length of a curve, (b) by using the arc length function.

Solution 7. Compute $\mathbf{r}'(t) = (t, \cos t, -\sin t)$. Then

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} |\mathbf{r}'(t)|\,dt \\ &= \int_0^{\frac{\pi}{4}} \sqrt{t^2+1}\,dt \\ &= \frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\sinh^{-1} t\Big|_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{32}\sqrt{\pi^2+16} + \frac{1}{2}\sinh^{-1}\frac{\pi}{4}. \end{aligned}$$

The arc length function is given by

$$\begin{aligned} s(t) &= \int_0^t |\mathbf{r}'(\tau)|\,d\tau \\ &= \int_0^t \sqrt{\tau^2+1}\,d\tau \\ &= \frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\sinh^{-1} t. \end{aligned}$$

Then

$$L = s(\frac{\pi}{4}) - s(0) = \frac{\pi}{32}\sqrt{\pi^2+16} + \frac{1}{2}\sinh^{-1}\frac{\pi}{4}.$$